Mathematical Aspects of Quantum Field Theory

Michael Keyl (FU Berlin)[1]

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[1]Mail adress: michaelkeyl137@gmail.com
[2]Mail adress: frederik.vom-ende@tum.de

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0 Introduction

In quantum mechanics, the standard approach is as follows. We have a Hilbert space $\mathcal{H}$ (e.g. $\mathcal{H} = L^2(\mathbb{R}^3)$ for a particle in three dimensions), a Hamiltonian H describing the dynamics (like a Schrödinder operator $H = -\Delta + V(x)$ for a particle moving in a potential V), and the standard observables position Q_j and momentum P_k described by self-adjoint operators on $\mathcal{H}$. In Quantum Field Theory very loosely speaking, the main difference concerns these observables. While we still work on Hilbert spaces and with Hamiltonians, the Q_j and P_k are replaced by field amplitudes $\mathbb{R}^3 \ni x \mapsto \phi(x)$ and field momenta $\mathbb{R}^3 \ni x \mapsto \pi(x)$ at a point (or space time event) x. Mathematically they are usually not described by operators but quadratic forms in $\mathcal{H}$.

There are many reasons to use fields rather than particles. Often both discriptions are equivalent and the corresponding fields admit an interpretation in terms of particles. In such a situation it is a matter of convenience which pictures should be preferred. But in some cases the particle point of view (or more generally spoken: the point of view of ordinary Quantum Mechanics) is too limited and fields are really needed for a consistent description. The most prominent example where this happens is *Relativistic* Quantum Mechanics which does not exist *as a consistent theory*. Problematic are in particular the following issues:

- *Locality*. Consider particle described by a wave function $\Psi \in L^2(\mathbb{R}^3)$ and located in the region Σ, i.e. $\operatorname{supp}\psi \subset \Sigma$. If we follow the usual rule and describe the probability to detect a particle in $\Sigma \subset \mathbb{R}^3$ by $\int_\sigma |\Psi(x)|^2\,dx$ the particle is located in Σ with certainty. After evolving the wave function freely, and for an arbitrarily short time ϵ (e.g. with the relativistic Hamilton operator $H = \sqrt{P^2 + m^2}$) the new wave function ψ_ϵ has non compact support, i.e. the probability to find the particle arbitrarily far away from its original position is non-zero; cf. Fig. 1. Something like this can also happen in non-relativistic models. In those cases, however, infinite speed is not a conceptual problem. In Special Relativity, on the other hand, it is.

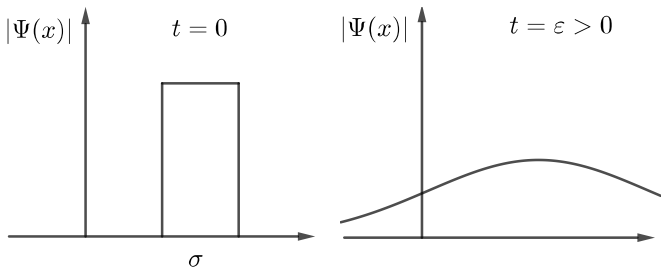

Figure 1: Possible evolution of the amplitude of the wave function for some arbitrary timestep $\varepsilon > 0$. This however is non-consistent with special relativity, in particular the speed of light.

- *Particle creation*. Even in simple setups involving relativistic particles in external potential particle creation effects can occur. If you want to describe this within

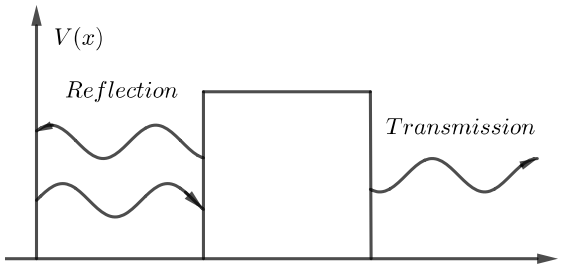

Figure 2: Klein's result showed that if the potential is of the order of the electron mass, the barrier is nearly transparent. Also for the dirac equation, more particles can be reflected than were sent in because of pair production. The number of particles not being fix cannot be described by quantum mechanics.

a model which is based on the assumption that the particle number is fixed this leads to contradictions. A typical example is Klein's paradox where an electron beam hitting a potential barrier is described. Within relativistic Quantum Mechanics we get strange behaviours of reflection and transmission coefficients which can be explained if we involve pair creation; cf. Fig. 2. Note that dynamical changes of the particle number can also occur in non-relativistic models (e.g. in solid state physics). However, in the relativistic case particle creation and annihilation is the rule even in simple cases involving only external potentials and no real interaction. It can be avoided only if we restrict our attention to free particles.

Both problems can be resolved by fields: Firstly, in Quantum Field Theory we can localize fields (observables) rather than wave functions (states), and secondly, fields are perfectly capable to describe particle models where the particle number is not fixed.

1 Wightman Quantum Field Theory

1.1 Tempered Distributions

For this section we orient ourselves towards [RS80, Chapter V]. First we introduce the following standard notation. For $f : \mathbb{R}^n \to \mathbb{C}$, $\alpha, \beta \in \mathbb{N}_0^n$ and $\mathbb{R}^n \ni x = (x_1, \ldots, x_n)$ we denote

$$x^\beta = x_1^{\beta_1} \ldots x_n^{\beta_n} \qquad |\alpha| = \sum_{j=1}^{n} \alpha_j \qquad D^\alpha f = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}}.$$

Definition 1.1. *The functions of rapid decrease, also called Schwartz functions, is the set defined as*

$$\mathscr{S}(\mathbb{R}^n) := \{f \in C^\infty(\mathbb{R}^n, \mathbb{C}) \,|\, \|f\|_{\alpha\beta} < \infty \quad \forall_{\alpha,\beta \in \mathbb{N}_0^n}\}$$

where $\|f\|_{\alpha\beta} = \sup_{x \in \mathbb{R}^n} |x^\alpha D^\beta f(x)|$.

Remark 1.2 (Locally compact spaces)**.** Let V be a complex vector space V and $(p_j)_{j \in I}$ a family of seminorms[3] which separates points, so $p_j(f) = 0$ for all $j \in I$ implies $f = 0$. Then $(V, (p_j)_{j \in I})$ is said to be a locally convex space. A neighborhood base at $0 \in V$ is given by the sets

$$\mathcal{N}(j_1, \ldots, j_m; \varepsilon) = \{f \in V \,|\, p_{j_k}(f) < \varepsilon \quad \forall_{k \in \{1,\ldots,m\}}\},$$

which can be regarded as an analogon of the family of ε-balls in Banach spaces. By translating the $\mathcal{N}(j_1, \ldots, j_m; \varepsilon)$ within V we can define similar bases at each $v \in V$. In this way V becomes a topological space, and addition and scalar multilpication become continous maps. In other words, V equipped with topology is a topological vector space. If $J = \mathbb{N}$, then we can define a metric d via

$$d(f, g) = \sum_{n=1}^{\infty} 2^{-n} \frac{p_n(f - g)}{1 + p_n(f - g)}$$

which generates the same topology as before. If $(V, (p_n)_{n \in \mathbb{N}})$ is complete, then it is called Fréchet space.

The following is an easy application of the neighbourhood definition given above.

Lemma 1.3. *A linear functional $\phi : V \to \mathbb{C}$ is continous if and only if one can find a finite set $j_1, \ldots, j_m \in I$ of seminorms and $C \in \mathbb{R}_0^+$ such that*

$$|\phi(x)| \leq C(p_{j_1}(x) + \ldots + p_{j_m}(x))$$

for each $x \in V$.

[3] A seminorm is a norm which is not definite so for $f, g \in V$ and $\lambda \in \mathbb{C}$ we have $p(\lambda f) = |\lambda| p(f)$ and $p(f + g) \leq p(f) + p(g)$, but $p(f) = 0$ does not necessarily imply $f = 0$.

Proposition 1.4 ([RS80], Theorem V.9). *The vector space $\mathscr{S}(\mathbb{R}^n)$ with the natural topology given by the seminorms $\|\cdot\|_{\alpha,\beta}$ is a Fréchet space.*

Definition 1.5. *The topological dual[4] $\mathscr{S}'(\mathbb{R}^n)$ of $\mathscr{S}(\mathbb{R}^n)$ is called the space of tempered distributions.*

Example 1.6. 1. (δ-distribution). Consider $\delta_x : \mathscr{S}(\mathbb{R}^n) \to \mathbb{C}$ for some $x \in \mathbb{R}^n$ where $f \mapsto \delta_x(f) := f(x)$. We then have

$$|\delta_x(f)| \leq \sup_{y\in\mathbb{R}^n} |f(y)| = \|f\|_{0,0},$$

so $\delta_x \in \mathscr{S}'(\mathbb{R}^n)$. Sometimes it is useful to use the formal expression

$$f(x) = \int_{\mathbb{R}^n} \delta(x-y) f(y)\, dy$$

which involves the "delta-function" δ. When we do this it is important to keep in mind that δ really does not exist as a function, i.e. we can not evaluate it at each $x \in \mathbb{R}$, we can only evaluate it "under the integral".

2. (Measures). Consider a finite Borel measure μ, so we can define $\mu : \mathscr{S}(\mathbb{R}^n) \to \mathbb{C}$, $f \mapsto \int_{\mathbb{R}^n} f(x)\mu(dx)$. Continuity of this map can be shown analogously to the δ-distribution case, so $\mu \in \mathscr{S}'(\mathbb{R}^n)$.
3. Let us look at this special case of the second example. For $g \in \mathscr{S}(\mathbb{R}^n)$ we can define $\phi_g(f) = \int_{\mathbb{R}^n} f(x)g(x)\, dx$. Moreover, if $g_1 \neq g_2$ as functions in $\mathscr{S}$, then $\phi_{g_1} \neq \phi_{g_2}$. This embeds $\mathscr{S}$ naturally in $\mathscr{S}'$.
4. Similarly, for $g \in L^p(\mathbb{R}^n)$ and $p \in \mathbb{N}$ we have

$$\phi_g(f) = \int_{\mathbb{R}^n} f(x)g(x)\, dx$$

which embeds $L^p \hookrightarrow \mathscr{S}'$ in a similar way.

Remark 1.7. We equip $\mathscr{S}'(\mathbb{R}^n)$ with the weak-$\star$-topology, which is generated by seminorms $f \mapsto |\phi(f)|$ with $\phi \in \mathscr{S}'(\mathbb{R})$. Then, the linear subset $\mathscr{S}(\mathbb{R}^n) \subset \mathscr{S}'(\mathbb{R}^n)$ is dense and the embedding $\iota : \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}'(\mathbb{R}^n)$ is continous. This suggests extending continous maps $T : \mathscr{S} \to \mathscr{S}$ to $\mathscr{S}'$ as follows. If $T : \mathscr{S} \to \mathscr{S}$ is continous, then $\iota \circ T : \mathscr{S} \to \mathscr{S}'$ is continous as well by continuity of ι. Since $\mathscr{S}$ is dense is $\mathscr{S}'$, there is at most one continous extension of $\iota \circ T$. To find this extension we look for a continous $S : \mathscr{S} \to \mathscr{S}$ with the adjoint $S' : \mathscr{S}' \to \mathscr{S}'$, $\phi \mapsto S'(\phi)$ and ask S' to satisfy $S'(\phi) = \phi \circ S$. Now S' is well-defined and continous in the weak-$\star$-topology. Hence S' is the extension we are looking for and it can be expressed as $T\phi(f) = \phi(S(f))$. In other words the general strategy is to apply the adjoint of T to the test function (assuming that the double adjoint becomes T again).

[4] The topological dual is the space of continous linear functionals acting on the vector space.

Example 1.8. 1. Consider a C^∞ function $F : \mathbb{R}^n \to \mathbb{C}$ which derivatives are polynomially bounded, so there exists $C \in \mathbb{R}_0^+$ and $n \in \mathbb{N}$, such that

$$|D^\alpha F(x)| \leq C(1 + \|x\|^2)^n$$

for all $\alpha \in \mathbb{N}_0^n$. This means for any $f \in \mathscr{S}(\mathbb{R}^n)$ we have $Ff \in \mathscr{S}(\mathbb{R}^n)$ where $f \mapsto Ff$ is continous. The extension in this case then is given by $(F\phi)f = \phi(Ff)$. Then

$$\phi_g(Ff) = \int_{\mathbb{R}^n} g(x)(F(x)f(x))\,dx = \int_{\mathbb{R}^n} (g(x)F(x))f(x)\,dx = \phi_{Fg}(f)$$

2. Weak derivative. To extend D^α to $\mathscr{S}'$, partial integration implies that $(D^\alpha\phi)(f) = (-1)^{|\alpha|}\phi(D^\alpha f)$.
3. For the Fourier transform, we simply have $\hat{\phi}(f) = \phi(\hat{f})$.
4. $f(\cdot) \to f(\cdot - a)$ translation, $f(\cdot) \to f(A\cdot)$ with $A \in \mathrm{GL}(n, \mathbb{R})$

Example 1.9 (Heaviside function)**.** Defining

$$\nu(x) = \begin{cases} x & x \geq 0 \\ 0 & x \leq 0 \end{cases},$$

we get the Heaviside function via

$$\Big(\frac{d}{dx}\phi_\nu\Big)(f) = -\phi_\nu(f') = -\int_0^\infty xf'(x)\,dx = \int_0^\infty f(x)\,dx,$$

which implies $\frac{d}{dx}\phi_\nu = \phi_\theta$, where

$$\theta(x) = \begin{cases} x & 1 \geq 0 \\ 0 & x \leq 0 \end{cases}.$$

Further

$$\Big(\frac{d}{dx}\phi_\theta\Big)(f) = -\phi_\theta(f') = -\int_0^\infty f'(x)\,dx = f(0)$$

so $\frac{d}{dx}\phi_\theta = \delta_0$.

Theorem 1.10 (Regularity theorem for distributions, [RS80] Theorem V.10)**.** *Let $\phi \in \mathscr{S}'(\mathbb{R}^n)$. Then $\phi = D^\beta g$ for some polynomially bounded continous function $g : \mathbb{R}^n \to \mathbb{C}$ and some $\beta \in I_+^n$, that is,*

$$\phi(f) = \int (-1)^{|\beta|} g(x)(D^\beta f)(x) d^n x$$

for all $f \in \mathscr{S}(\mathbb{R}^n)$.

Definition 1.11 (Support). *We say $\phi \in \mathscr{S}'(\mathbb{R}^n)$ vanishes on an open subset $\sigma \subset \mathbb{R}^n$ if $\phi(f) = 0$ for all f with $\operatorname{supp}(f) \subset \sigma$. Then $\operatorname{supp}\phi$ is the complement of the largest open set on which ϕ vanishes.*

Definition 1.12. *Let $\phi \in \mathscr{S}'(\mathbb{R}^n)$. We say $x \in \mathbb{R}^n$ is a regular point of ϕ if there exists an open neighbourhood U of x and $g \in C^\infty(U)$, such that $\phi(f) = \phi_g(f)$ for all f with $\operatorname{supp} f \subset U$. The complement of the set of regular points is called the singular support.*

An easy example is the singular support of the δ-distribution, which obviously only is the zero.

Theorem 1.13 (Kernel or nuclear theorem, [RS80] Theorem V.12). *Let $B(f,g)$ be a separately continous bilinear functional on $\mathscr{S}(\mathbb{R}^n) \times \mathscr{S}(\mathbb{R}^m)$. Then there is a unique tempered distribution $T \in \mathscr{S}'(\mathbb{R}^{n+m})$ with $B(f,g) = T(f \otimes g)$ where*

$$(f \otimes g)(x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+m}) = f(x_1, \ldots, x_n) g(x_{n+1}, \ldots, x_{n+m}).$$

The result can easily be extended to more than two tensor factors.

1.2 Quantum Fields

Let us, as a reminder, quickly go over some notation and results regarding unbounded operators. For this, we orient ourselves towards [RS80, Chapter VIII]. Here, $\mathcal{H}$ is any separable Hilbert space.

- An operator is a linear map $A : D(A) \to \mathcal{H}$. The subspace $D(A) \subset \mathcal{H}$ is the domain of A. If $D(A)$ is dense, then the operator is called densely defined.
- The graph of A is defined to be $\Gamma(A) = \{(x, Ax) \,|\, x \in D(A)\} \subset \mathcal{H} \times \mathcal{H}$. Then A is called closed if $\Gamma(A) \subset \mathcal{H} \times \mathcal{H}$ is closed.
- An operator B is said to be an extension of A if $\Gamma(A) \subset \Gamma(B)$, we then write $A \subset B$.
- An operator A is closable if it has a closed extension. If A is closable, it has a smallest closed extension $\overline{A}$ characterized by $\Gamma(\overline{A}) = \overline{\Gamma(A)}$. We then say $\overline{A}$ is the closure of A.
- Let A be densely defined. We define

  $$D(A^*) = \{x \in \mathcal{H} \,|\, \exists_{y \in \mathcal{H}} \forall_{z \in D(A)} : \langle Az, x\rangle = \langle z, y\rangle\}$$

  and for $x \in D(A^*)$ we define the adjoint via $A^* x := y$ for respective y. Note that $D(A^*) = \{0\}$ is possible.
- Let A be densely defined. Then A is closable if and only if $D(A^*)$ is densely defined. In that case, A^* is closed and the double adjoint satisfies $A^{**} = \overline{A}$.
- A is said to be symmetric or hermitian if $\langle Ax, y\rangle = \langle x, Ay\rangle$ for all $x, y \in D(A)$. This is equivalent to $D(A) \subset D(A^*)$ with $Ax = A^* x$ for all $x \in D(A)$. Also, A is called self-adjoint if $A = A^*$. In particular, this means $D(A) = D(A^*)$.

- If A is symmetric, then $A \subset A^{**} \subset A^*$. If A is symmetric and closed, then $A = A^{**} \subset A^*$. If A is self-adjoint, then $A = A^{**} = A^*$.
- We call A essentially self-adjoint if A is symmetric and $\overline{A}$ is self-adjoint.

For the topic of quantum fields, we refer to [RS75, Chapter IX.8].

Definition 1.14. *A quantum field is a 4-tuple $(\mathcal{H}, D, \Phi, \Omega)$ consisting of a (separable) Hilbert space $\mathcal{H}$, a dense subspace $D \subset \mathcal{H}$, a $\mathbb{C}$-linear map*

$$\Phi : \mathscr{S}(\mathbb{R}^n) \to \mathcal{L}(D, \mathcal{H}),$$

where $\mathcal{L}(D, \mathcal{H})$ denotes the linear maps between D and $\mathcal{H}$, and an element $\Omega \in D \subset \mathcal{H}$, such that the following conditions hold.

(a) D is invariant, so $\Phi(f)D \subset D$ for all $f \in \mathscr{S}(\mathbb{R}^n)$.

(b) Ω is cyclic which means that the set

$$D_0 := \{\Phi(f_1) \dots \Phi(f_m)\Omega \,|\, f_1, \dots, f_m \in \mathscr{S}(\mathbb{R}^n), m \in \mathbb{N}\} \subset D \subset \mathcal{H}$$

is dense in $\mathcal{H}$.

(c) $\Phi(f)$ is closable for all $f \in \mathscr{S}(\mathbb{R}^n)$.

(d) For all $x, y \in D$, the map

$$\mathscr{S}(\mathbb{R}^n) \ni f \longmapsto \langle x, \Phi(f)y \rangle$$

is a tempered distribution.

Note that D_0 can replace D since D_0 is also dense in $\mathcal{H}$.

Definition 1.15. *A quantum field $(\mathcal{H}, D, \Phi, \Omega)$ is called hermitian if*[5]

$$\Phi(\overline{f}) = \Phi(f)^*|_D$$

and if for $f \in \mathscr{S}(\mathbb{R}^n, \mathbb{R})$ the operator $\Phi(f)$ is essentially self-adjoint on D.

Remark 1.16 (Interpretation)**.** A hermitian quatum field describes a special observable, or more precisely a whole family of observables. To understand this remark a bit better let us pretend that we can write $\Phi(f)$ as

$$\Phi(f) = \int_{\mathbb{R}} f(x)\Phi(x)\, dx \tag{1.1}$$

with self-adjoint operators $\Phi(x)$, $x \in \mathbb{R}$. Note that this is usually not possible and we will discuss in Ch. 2 how Eq. (1.1) can be interpreted in a mathematical rigorous way. For now we just look at the operator-valued field $\Phi(x)$ as a formal expression for an operator-valued distribution in the same way as the delta-function $\delta(x)$ is a formal expression

[5] $\overline{f}$ denotes the complex conjugate of f

for the delta-distribution δ_0. In doing so we can look at $\Phi(x)$ as the observable which measures the *field amplitude* in $x \in \mathbb{R}^n$. The smeared out version $\Phi(f)$ from Eq. (1.1) can then be regarded as the *averaged* field amplitude with averaging function f. This interpretation makes it very clear that the $\Phi(f)$ are *local* observables, i.e. they can be measured in any region containing the support of f.

To keep this interpretation, self-adjointness of $\Phi(f)$ for real-valued f is mandatory. Mathematically, however, this self-adjointness condition is often annoying, since it is difficult to prove. Most others drop it therefore and accept hermitian quantum fields where the $\Phi(f)$ are really only hermitian. We will see in the next Section why this can be advantageous.

Remark 1.17. Up to now we can generalize everything to manifolds by replacing $\mathscr{S}(\mathbb{R}^n)$ with

$$D(M) = \{f : M \to \mathbb{C} \,|\, f \text{ is } C^\infty \text{ and compactly supported}\}$$

for some C^∞-manifold M.

Remark 1.18 (Poincaré group)**.** The definitions presented so far are fairly general and lack in particular any dynamical content. This will change now, when we study quantum fields in Minkowski space. For that, let us have a short recap on some concpets and notations.

- The Minkowski metric on $\mathbb{R}^4$ is given by

$$\eta(v, w) = v^0 w^0 - \sum_{j=1}^{3} v^j w^j.$$

- The Lorentz group is defined to be

$$O(3,1) = \{\Lambda \in \mathrm{GL}(4, \mathbb{R}) \,|\, \eta(\Lambda v, \Lambda w) = \eta(v, w)\}.$$

  The restricted Lorentz group then is

$$\mathrm{SO}^\uparrow(1,3) = \{\Lambda \in O(3,1) \,|\, \det \Lambda = 1, \langle e_0, \Lambda e_0 \rangle > 0\}.$$

- The restricted Poincaré group $\mathcal{P}_+^\uparrow$ is the semi-direct product $\mathbb{R}^4 \rtimes \mathrm{SO}^\uparrow(1,3)$. Hence $\mathcal{P}_+^\uparrow$ is the set of pairs (a, Λ) where $a \in \mathbb{R}^4$ and $\Lambda \in \mathrm{SO}^\uparrow(1,3)$, and the group operation is

$$(a, \Lambda_1)(b, \Lambda_2) = (a + \Lambda_1 b, \Lambda_1 \Lambda_2).$$

  The Poincaré group describes transformations from one inertial system into another by the coordinate transformation $v \mapsto a + \Lambda v$.

- Now assume that $\mathcal{P}_+^\uparrow \ni (a, \Lambda) \mapsto U(a, \Lambda) \in \mathcal{U}(\mathcal{H})$, with the unitary group $\mathcal{U}(\mathcal{H})$ on

$\mathcal{H}$, is a strongly continuous[6], unitary representation of $\mathcal{P}_+^\uparrow$ on $\mathcal{H}$. By strong continuity we can define generators of the translations

$$P_j\xi = -i\,\frac{d}{d\lambda}U(\lambda e_j, \mathbb{1})\xi\Big|_{\lambda=0}$$

for $j = 0, \ldots, 3$ with standard basis $(e_j)_j$ of $\mathbb{R}^4$. Here $\xi \in \mathcal{H}$ is chosen such that the limit $\lambda \to 0$ exists. Note that $U(\lambda e_j, \mathbb{1})\xi$ is a time translation for $j = 0$ and a space translation if $j = 1, 2, 3$. Hence P_0 is the Hamiltonian and $P_1, \ldots, P_3$ are momentum operators. With an arbitrary $a \in \mathbb{R}^4$ we get

$$U(a, \mathbb{1}) = \exp\Big(i\sum_{j=0}^{3} a_j P_j\Big). \tag{1.2}$$

- If $a = \Lambda e_0$ with a Lorentz transfomation Λ, the generator P_a of the one parameter group $\mathbb{R} \ni t \mapsto U(ta, \mathbb{1}) \in \mathcal{U}(\mathcal{H})$ is given by

$$P_a = U(0, \Lambda)P_0U(0, \Lambda)^* = \sum_{j=0}^{3} a_j P_j$$

  In other words, P_a is the Hamiltonian the inertial observer with four-velocity a is seeing. The Lorentz transformation Λ describes the transition from "our" intertial system (i.e. where we are at rest) into the one of observer a. Please note that according to this reasoning the representation $U(a, \Lambda)$ does not only describes the transition between inertial systems, but also contains the *complete description of the dynamical structure of our theory.*

- Since the translations form an abelian subgroup, the unitaries $U(a, \mathbb{1})$ mutually commute and can therefore be jointly "diagonalized". More precisely, by the spectral theorem ([RS80], Ch. VII and Sec. VIII.3), there is a projection valued measure[7]

$$E : \mathfrak{B}(\mathbb{R}^4) \to \mathcal{B}(\mathcal{H}) \qquad \Sigma \longmapsto E(\Sigma)$$

  on $\mathbb{R}^4$ such that

$$\langle \xi, U(a, \mathbb{1})\xi\rangle = \int_{\mathbb{R}^4} \exp\Big(i\sum_j a_j\lambda_j\Big)\,\langle \xi, E(d\lambda)\xi\rangle. \tag{1.3}$$

- Readers which are unfamiliar with measure theory should think of $\langle \xi, E(d\lambda)\xi\rangle$ as $E_\xi(\lambda)\,d\lambda$ with singular (i.e. "distributional") density function $E_\xi(\lambda)$. The latter can be written as $E_\xi(\lambda) = \langle \xi, E(\lambda)\xi\rangle$ with a likewise singular map E from $\mathbb{R}^4$ into the set of projection operators in $\mathcal{H}$. Note, however, that this is a very handwaving point of view. A mathematical rigorous interpretation *requires* measure theory.

[6] This means that $\mathcal{P}_+^\uparrow \ni (\alpha, \Lambda) \mapsto U(\alpha, \Lambda)\xi \in \mathcal{H}$ is continous for all $\xi \in \mathcal{H}$.

[7] Here, $\mathfrak{B}(\mathbb{R}^4)$ denotes the σ-algebra of Borel subsets of $\mathbb{R}^4$ and $\mathcal{B}(\mathcal{H})$ are the bounded operators on $\mathcal{H}$.

- Finally we define two regions σ_1, σ_2 to be *spacelike separated* if we can not reach σ_1 from σ_2 with a causal curve[8] and vice versa. This is best described in a picture; cf. Figure 3.

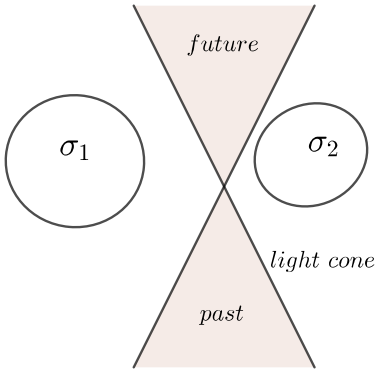

Figure 3: Two open regions $\sigma_1, \sigma_2 \subset \mathbb{R}^4$ are called spacelike separated if they are separated by the light cone so there is no physically possible information exchange between them. Formally this means that for any $x \in \sigma_1$, $y \in \sigma_2$ we have $\eta(x-y, x-y) < 0$.

Definition 1.19. *A 5-tuple $(\mathcal{H}, D, \Phi, \Omega, U)$ consisting of a hermitian quantum field $(\mathcal{H}, D, \Phi, \Omega)$ and a strongly continous representation of the Poincaré group U is called Wightman quantum field if the following conditions are fulfilled.*

(a) (Local commutativity or microscopic causality). If f and g in $\mathscr{S}(\mathbb{R}^4)$ have supports which are spacelike separated, then $[\Phi(f), \Phi(g)] = 0$.

(b) (Special covariance). For each $(a, \Lambda) \in \mathcal{P}_+^\uparrow$ and all $f \in \mathscr{S}(\mathbb{R}^4)$ we have

$$\Phi((a,\Lambda), f) = U(a,\Lambda)\Phi(f)U(a,\Lambda)^*,$$

where $((a,\Lambda), f)(x) = f(\Lambda^{-1}(x-a))$.

(c) (Uniqueness and invariance of the vacuum). There exists a unique vector $\Omega \in \mathcal{H}$ such that for all $a \in \mathbb{R}^4$ we have

$$U(a, \mathbb{1})\Omega = \Omega.$$

(d) (Spectral condition). The support of the spectral measure E from Eq. (1.3) is contained in the forward light cone $V^+ = \{v \in \mathbb{R}^4 \,|\, V_0 > 0,\ \eta(v,v) > 0\}$.

Remark 1.20. The local commutativity condition mathematically expresses the quantum-mechanical statement that measurements in spacelike separated regions should be jointly measurable; cf. the discussion in Remark 1.16. Note that commutation in the given form does not guarantee jointly measurability, but it is a necessary condition; cf. in this context [RS80], Sec. VIII.5.

[8] This means by travelling with at most the speed of light.

Remark 1.21 (Spectrum condition)**.** The spectral condition means that the joint spectrum of P_j is contained in the forward light cone, so $\sigma(P_0) \subset \mathbb{R}_0^+$ and the energy is positive for all inertial systems. This can be seen very easily and without advanced knowledge if we assume that all the P_j have purely discrete spectrum. Note that this is not very realistic from the physical point of view since the four momenta usually have continuous spectrum and the three-momentum typically has no eigenvalues at all. It is, however, a very simple case where measure theory is not required and therefore it can help to understand the role of the spectral measure E and the spectrum condition.

Since the P_j are mutually commuting, there exists a complete orthonormal system $(\phi_n)_{n\in\mathbb{N}}$ in $\mathcal{H}$ with $P_j\phi_n = \lambda_{jn}\phi_n$ for $j = 0, \dots, 3$. Defining $\lambda_n = (\lambda_{0n}, \dots, \lambda_{3n})$ we get

$$\exp\Big(i\sum_{j=0}^{3} a_j P_j\Big)\phi_n = \exp(ia\cdot\lambda_n)\phi_n \implies \exp\Big(i\sum_{j=0}^{3} a_j P_j\Big) = \sum_{n=0}^{\infty} e^{ia\cdot\lambda_n}|\phi_n\rangle\langle\phi_n|.$$

where the sum is strongly convergent[9]. With this and (1.2), we see that the matrix elements of the unitaries are given by

$$\langle x, U(a,\mathbb{1})x\rangle = \sum_{n=0}^{\infty} e^{ia\cdot\lambda_n}|\langle x,\phi_n\rangle|^2.$$

Written as an integral, we get

$$\langle x, U(a,\mathbb{1})x\rangle = \int_{\mathbb{R}^4} e^{ia\cdot\lambda} \underbrace{\sum_{n=0}^{\infty}\delta(\lambda-\lambda_n)|\langle x,\lambda_n\rangle|^2}_{E_x(\lambda)}\, d\lambda.$$

Another option is to use $E_\lambda = \sum_{n=0}^{\infty}\delta(\lambda-\lambda_n)|\phi_n\rangle\langle\phi_n|$ which yields

$$\langle x, U(a,\mathbb{1})x\rangle = \int_{\mathbb{R}^4} e^{ia\cdot\lambda} d\langle x, E_\lambda x\rangle.$$

This expression also makes sense in the non-discrete case. The support of the spectral measure E obviously is given by

$$\sigma(E) = \{\lambda_n \,|\, n\in\mathbb{N}\} \subset \mathbb{R}^4.$$

The spectral condition now demands

$$\sigma(E) \subset V_+ = \{v\in\mathbb{R}^4 \,|\, \eta(v,v) > 0, v^0 > 0\},$$

where V_+ is the forward light cone. Now $\lambda_n \in V_+$ by definition means $\lambda_{0n} > 0$, which implies positive energy.

[9] This means that $\sum_{n=0}^{\infty} e^{ia\cdot\lambda_n}|\phi_n\rangle\langle\phi_n,\xi\rangle$ converges for all $\xi\in\mathcal{H}$

Remark 1.22 (Interpretation)**.** A Wightman quantum field describes a physical system which transforms covariantly under Poincaré tranformations (i.e. change of inertial systems). It combines the unitary representation $U(a, \Lambda)$ of the Poincaré group, and all objects and concepts derived from it, like the four-momentum operators P_j, $j = 0, \ldots, 3$ (cf. Remark 1.18), with a Hermitian quantum field $\Phi(f)$. The P_j contain in particular the dynamical description of the model. As observables they are not that important from a practical point of view, since they are *global*, i.e. they measure the four-momentum of the whole universe. The field operators $\Phi(f)$ on the other hand are of *local* nature, as pointed out in Remark 1.16. Hence they are more realisitically linked to quantities which actually *can* be measured in an experiment. A typical model involving quantum fields usually contains more than one field, describing different physical quantities or observables. For example in addition to the field Φ describing field amplitudes we might want to look at components of the energy momentum tensor, which are described by additional fields within the same model.

The previous remark might create the impression that the fields are completely kinematical objects, while the dynamics is exclusively contained in the representation $U(a, \Lambda)$. This point of view, however, is wrong. Since a Wightman field is a spacetime field, it does contain dynamical information. The axioms in Definition 1.19, in particular the Poincaré covariance and the invariance of the vacuum, link the field Φ and the representation $U(a, \Lambda)$ very closely together, in other words they are not independent, and the fields contain informations about the representation $U(a, \Lambda)$.

Remark 1.23 (Wightman axioms)**.** We haven't explicitly talked about the Wightman axioms for a scalar field, because we have distributed them over Definitions 1.14, 1.15 and 1.19. In Def. 1.14 and 1.15 we find *(numerations taken from [RS75], Sec. IX.8)*: Invariant domain for the fields (Axiom 4), regularity of the fields (Axiom 5) and the cyclicity of the vaccuum (Axiom 8). In Definition 1.19 we have the existence of the representation U ("relativistic invariance of states"; Axiom 1), the spectral condition (Axiom 2), the invariance of the vacuum ("existence and uniqueness of the vaccum"; Axiom 3), the Poincaré invariance of the fields (Axiom 6) and the local commutativity (Axiom 7).

1.3 Wightman Distributions

Consider a hermitian quantum field $(\mathcal{H}, D, \Phi, \Omega)$ and define

$$\tilde{W}^{(m)} : \underbrace{\mathscr{S}(\mathbb{R}^n) \times \ldots \times \mathscr{S}(\mathbb{R}^n)}_{m \text{ factors}} \to \mathbb{C} \qquad (f_1, \ldots, f_m) \longmapsto \langle \Omega, \Phi(f_1) \ldots \Phi(f_m) \Omega \rangle$$

for any $m \in \mathbb{N}$. By regularity of the field, $\tilde{W}^{(m)}$ is separately continous in all arguments because

$$f_1 \longmapsto \langle \underbrace{\Omega}_{\in D}, \Phi(f_1) \underbrace{\ldots \Phi(f_m)\Omega}_{\in D} \rangle$$

is continous, the same is true for

$$f_2 \longmapsto \langle \underbrace{\Phi(f_1)^*\Omega}_{\in D}, \Phi(f_2) \underbrace{\ldots \Phi(f_m)\Omega}_{\in D} \rangle$$

and so forth. By the nuclear theorem *(see Theorem 1.13)*, there exists a unique distribution $W^{(m)} \in \mathscr{S}'(\mathbb{R}^n \times \ldots \times \mathbb{R}^n) = \mathscr{S}'(\mathbb{R}^{n \cdot m})$ such that

$$W^{(m)}(f_1 \otimes \ldots \otimes f_m) = \tilde{W}^{(m)}(f_1, \ldots, f_m).$$

In particular, $\tilde{W}^{(m)}$ is jointly continous.

Definition 1.24. *The $W^{(m)}$, $m \in \mathbb{N}$ are called Wightman distributions of the quantum field $(\mathcal{H}, D, \Phi, \Omega)$.*

The tasks now are to reconstruct a quantum field from $W^{(m)}$ and to translate the Wightman axioms into conditions on $W^{(m)}$. The solution to both tasks is known as the Wightman reconstruction theorem

Definition 1.25. *A complex vector space $\mathcal{A}$ is called unital $*$-algebra if $\mathcal{A}$ is equipped with a bilinear, associative product,*

$$\mathcal{A} \times \mathcal{A} \ni (A, B) \longmapsto AB \in \mathcal{A}$$

and an antilinear[10] involution[11] ($*$-operation)

$$\mathcal{A} \ni A \longmapsto A^* \in \mathcal{A}$$

which satisfies $(AB)^* = B^*A^*$ and there exists a unit $\mathbb{1} \in \mathcal{A}$ with $A\mathbb{1} = \mathbb{1}A = A$ for all $A \in \mathcal{A}$.

Definition 1.26. *A functional $\omega : \mathcal{A} \to \mathbb{C}$ is called a state of $\mathcal{A}$ if*

(a) ω is linear and continous.

(b) (Positivity). $\omega(A^*A) \geq 0$ for all $A \in \mathcal{A}$.

(c) (Normalization). $\omega(\mathbb{1}) = 1$.

Lemma 1.27. *A state ω of a $*$-algebra $\mathcal{A}$ has the following properties*

(a) (Symmetry). $\omega(A^*B) = \overline{\omega(B^*A)}$.

(b) (Cauchy-Schwarz). $|\omega(A^*B)|^2 \leq \omega(A^*A)\omega(B^*B)$.

Proof idea. Look at $\omega((\lambda A + B)^*(\lambda A + B)) \geq 0$. Otherwise cf. [BR02] Lemma 2.3.10. □

[10] This means $(A + \lambda B)^* = A^* + \overline{\lambda} B^*$.
[11] This means $A^{**} = A$.

Definition 1.28. *A cyclic representation of a $*$-Algebra is a 4-tuple $(\mathcal{H}, D, \pi, \Omega)$ consisting of a Hilbert space $\mathcal{H}$, a dense subspace $D \subset \mathcal{H}$, a vector $\Omega \in D$ and a complex linear map*

$$\pi : \mathcal{A} \to \mathcal{L}(D, D) \subset \mathcal{L}(D, \mathcal{H}),$$

such that the following holds for all $A, B \in \mathcal{A}$.

(a) $\pi(AB) = \pi(A)\pi(B)$.

(b) $\pi(A^*) = \pi(A)^*|_D$.

(c) (Cyclicity). $\{\pi(A)\Omega \,|\, A \in \mathcal{A}\} = D_0 \subset D \subset \mathcal{H}$ *is dense.*

Again, D_0 is automatically an allowed domain and actually the smallest one.

Theorem 1.29 (Gelfand–Naimark–Segal (GNS)-representation)**.** *Let $\mathcal{A}$ be $*$-algebra and $\omega : \mathcal{A} \to \mathbb{C}$ be a state. Then there exists a cyclic representation $(\mathcal{H}_\omega, D_\omega, \pi_\omega, \Omega_\omega)$, such that*

$$\omega(A) = \langle \Omega_\omega, \pi_\omega(A)\Omega_\omega \rangle \tag{1.4}$$

holds for all $A \in \mathcal{A}$. If $D_\omega = D_0$, this representation is unique up to unitary equivalence.

Proof. On $\mathcal{A}$ we define

$$\langle A, B \rangle = \omega(A^*B)$$

which is sesquilinear and positive semi-definite by Lemma 1.27, but $\omega(A^*A) = 0$ may happen for $A \neq 0$. Therefore we define

$$\mathscr{I}^\omega = \{A \in \mathcal{A} \,|\, \omega(A^*A) = 0\}.$$

We want to show that $\mathscr{I}^\omega$ is a left ideal in $\mathcal{A}$. First, $\mathscr{I}^\omega$ is linear for the following reason.

(a) For $A \in \mathscr{I}^\omega$, $\lambda \in \mathbb{C}$ now

$$\omega((\lambda A)^*(\lambda A)) = |\lambda|^2 \omega(A^*A)$$

implies $\lambda A \in \mathscr{I}^\omega$.

(b) For $A, B \in \mathscr{I}^\omega$ we have

$$\omega((A+B)^*(A+B)) = \underbrace{\omega(A^*A)}_{=0} + \underbrace{\omega(B^*B)}_{=0} + 2\,\mathrm{Re}\,\omega(A^*B) = 0.$$

It is easy to see that $2\,\mathrm{Re}\,\omega(A^*B)$ vanishes by Cauchy-Schwarz.

Now $A \in \mathscr{I}^\omega$ implies $|\langle A, B \rangle|^2 \leq \omega(A^*A)\omega(B^*B)$ so $\langle A, B \rangle = 0$ for all $B \in \mathcal{A}$ and thus

$$\mathscr{I}^\omega = \langle A \in \mathcal{A} \,|\, \omega(AB) = 0 \;\forall_{B \in \mathcal{A}} \}.$$

Finally, $\mathscr{I}^\omega$ is a left ideal because

$$\omega((BA)^*C) = \omega(A^*(B^*C)) = 0 \implies BA \in \mathscr{I}^\omega$$

for any $A \in \mathscr{I}^\omega$, $B, C \in \mathcal{A}$. Now we define $D_\omega = \mathcal{A}\backslash\mathscr{I}^\omega$ and a scalar product $\langle [A], [B] \rangle = \langle A, B \rangle$ which is well-defined as can be verified readily. Our Hilbert space $\mathcal{H}$ then is given by the completion of D_ω, which by construction is dense in $\mathcal{H}$. For $A, B \in \mathcal{A}$, $I \in \mathscr{I}^\omega$ we have

$$A(B+I) = AB + \underbrace{AI}_{\in \mathscr{I}^\omega} \implies A(B+I) \in [AB].$$

Thus we can define π_ω by

$$\pi_\omega(A)[B] = [AB],$$

as well as $\Omega_\omega = [\mathbb{1}]$. Then $\pi_\omega(A)\Omega = [A]$ and $\{\pi_\omega(A)\Omega \,|\, A \in \mathcal{A}\} = D_\omega \subset \mathcal{H}$ is dense. Finally, we have to check (1.4).

$$\langle \Omega_\omega, \pi_\omega(A), \Omega_\omega \rangle = \langle [\pi], \pi_\omega(A)[\pi] \rangle = \langle [\pi], [A\pi] \rangle = \langle [\mathbb{1}], [A] \rangle = \omega(\mathbb{1}A) = \omega(A)$$

Now for the uniqueness. Let $(\tilde{\mathcal{H}}, \tilde{D}, \tilde{\pi}, \tilde{\Omega})$ be another cyclic representation which satisfies (1.4) with $\tilde{D} = \tilde{D}_0 = \{\tilde{\pi}(A)\tilde{\Omega} \,|\, A \in \mathcal{A}\}$. Define

$$U : D_\omega \to \tilde{D} \qquad U\pi_\omega(A)\Omega_\omega = \tilde{\pi}(A)\tilde{\Omega}$$

After showing that U is well-defined and unitary, one extends it to $\mathcal{H}$ and sees that $UD_\omega = \tilde{D}$ and $U\pi_\omega(A)U^* = \tilde{\pi}(A)$. □

Proposition 1.30. *Consider finite direct sums*

$$\mathcal{A} = \mathbb{C} \oplus \mathscr{S}(\mathbb{R}^n) \oplus \mathscr{S}(\mathbb{R}^n \times \mathbb{R}^n) \oplus \dots$$

with seminorms $\|f\|^{(m)}_{\alpha\beta} = \|f^{(m)}\|_{\alpha\beta}$, *where*

$$f = f^{(1)} \oplus \dots \oplus f^{(m)} \oplus \dots \oplus f^{(n)}$$

and $f^{(n)} \in \mathscr{S}(\mathbb{R}^{n\cdot m})$, *and consider the product*

$$\begin{aligned} fg &= [f^{(0)} \oplus f^{(1)} \oplus \dots \oplus f^{(n)}] \otimes [g^{(0)} \oplus \dots \oplus g^{(k)}] \\ &= f^{(0)} \otimes g^{(0)} \oplus (f^{(0)} \otimes g^{(1)} + f^{(1)} \otimes g^{(0)}) \oplus (f^{(0)} \otimes g^{(2)} + f^{(1)} \otimes g^{(1)} + f^{(2)} \otimes g^{(0)}) \oplus \dots, \end{aligned}$$

as well as the $*$*-operation*

$$(f^*)^{(m)}(x_1, \dots, x_m) = \overline{f^{(m)}(x_m, \dots, x_1)}.$$

Then $\mathcal{A}$ is locally convex space and $*$-algebra. Further, the operations are continous so $\mathcal{A}$ is a topological $*$-algebra with unit $\mathbb{1} = 1 \oplus 0 \oplus 0 \oplus \ldots \oplus 0$.

Definition 1.31. *The $\mathcal{A}$ just defined is called Borchers-Uhlmann-algebra (BU-Algebra).*

The following proposition is a straightforward consequence of the definitions.

Proposition 1.32. *Let $(\mathcal{H}, D, \Phi, \Omega)$ be a hermitian quantum field with Wightman distributions $W^{(m)}$. The functional*

$$\omega(f^{(0)} \oplus f^{(1)} \oplus \ldots \oplus f^{(m)}) = \sum_{\alpha=0}^{m} W^{(\alpha)} f^{(\alpha)} \tag{1.5}$$

is a state of the BU-Algebra.

Proposition 1.33. *Let $(\mathcal{H}, D, \Phi, \Omega)$ be a hermitian quantum field. There is a unique cyclic representation $(\mathcal{H}, D, \pi, \Omega)$ such that*

$$\pi(f_1 \otimes \ldots \otimes f_m) = \Phi(f_1) \ldots \Phi(f_m).$$

Further, π is the GNS representation of ω from (1.5).

Proof idea. First, we define $\mathcal{A}_0 \subset \mathcal{A}$ generated by tensor products $f_1 \otimes \ldots \otimes f_m$ which again is a $*$-algebra. The representation π is the GNS representation of ω restricted to $\mathcal{A}_0$ by the definition of the Wightman distributions. Consider the GNS representation $(\mathcal{H}_\omega, D_\omega, \pi_\omega, \Omega_\omega)$ of $\mathcal{A}$ with respect to the state ω. If we restrict π_ω to $\mathcal{A}_0$ we get a new representation $\tilde{\pi}$. We show that Ω_ω is cyclic for $\tilde{\pi}$ by using the following facts:

(a) The span of tensor products $f_1 \otimes \ldots \otimes f_m$ is dense in $\mathscr{S}(\mathbb{R}^n \times \ldots \times \mathbb{R}^n)$ by the N representation theorem; cf. [RS80] Theorem V.13. Hence $\mathcal{A}_0$ is dense in $\mathcal{A}$.

(b) The representation is continous in the sense that

$$\mathscr{S}(\mathbb{R}^n \times \ldots \times \mathbb{R}^n) \ni f \longmapsto \langle \xi, \pi_\omega(f)\psi \rangle$$

is in $\mathscr{S}'(\mathbb{R}^{n \cdot m})$ (by the nuclear theorem) for all $\xi, \psi \in D_\omega$.

Assume that Ω_ω is not cyclic for $\tilde{\pi}$. Then there is a $0 \neq \xi \in \mathcal{H}_\omega$ such that $\langle \tilde{\pi}(f)\Omega_\omega, \xi \rangle = 0$ for all $f \in \mathcal{A}_0$. But since π_ω is the GNS representation of $\mathcal{A}$ with respect to ω, the vector Ω_ω is cyclic for π_ω, i.e. there is a $f \in \mathcal{A}$ such that $\langle \pi_\omega(f)\Omega_\omega, \xi \rangle \neq 0$. By the denseness of $\mathcal{A}_0$ in $\mathcal{A}$ there is a sequence of $f_k \in \mathcal{A}_0$ $k \in \mathbb{N}$ converging to $f \in \mathcal{A}$. Hence by the continuity just stated we get

$$\lim_{k \to 0} \langle \tilde{\pi}(f_k)\Omega_\omega, \xi \rangle = \lim_{k \to 0} \langle \pi_\omega(f_k)\Omega_\omega, \xi \rangle = \langle \pi_\omega(f)\Omega_\omega, \xi \rangle$$

in contradiction to $\langle \tilde{\pi}(f_k)\Omega_\omega, \xi \rangle = 0$ and $\langle \pi_\omega(f)\Omega_\omega, \xi \rangle \neq 0$. Hence Ω_ω is a cyclic vector for $\tilde{\pi}$ and the latter is therefore unitarily equivalent to the GNS representation of $\mathcal{A}_0$ with respect to ω. Hence there exists a unitary $U : \mathcal{H}_\omega \to \mathcal{H}$ such that

$$U\pi_\omega(f_1) \ldots \pi_\omega(f_m)U^* = \Phi(f_1) \ldots \Phi(f_m).$$

For arbitrary $f \in \mathscr{S}(\mathbb{R}^{n\cdot m})$ we define $\Phi(f) := U\pi_\omega(f)U^*$, which is the extension we are looking for. The uniqueness follows from the fact that the matrix elements $\langle \xi, \Phi(f)\psi\rangle$ for $\xi, \psi \in D_0$ are uniquely determined by ω. □

Remark 1.34 (Fields and representations)**.** The previous results show that a quantum field (H, D_0, Φ, Ω) can be recovered (up to unitary equivalence) from its Wightman distributions $W^{(m)}$. The $W^{(m)}$ define a state ω on the BU-algebra $\mathcal{A}$, this state has a GNS representation $(\mathcal{H}_\omega, D_\omega, \pi_\omega, \Omega_\omega)$ which by Proposition 1.32 is unitarily equivalent to the representation defined by Φ, i.e. there is a unitary U with $U\Phi(f_1)\cdots\Phi(f_m)U^* = \pi_\omega(f_1 \otimes \cdots \otimes f_m)$ for all $f_1, \ldots, f_m \in \mathscr{S}(\mathbb{R}^n)$ and all $m \in \mathbb{N}$. Hence we recover a unitarily equivalent copy of Φ by

$$\mathscr{S}(\mathbb{R}^n) \ni f \mapsto \Phi_\omega(f) = \pi_\omega(f) \in \mathcal{L}(D_\omega, D_\omega). \tag{1.6}$$

The last equation shows that we even have a little bit more. If ω is any continuous state of $\mathcal{A}$, it defines its GNS representation $(\mathcal{H}_\omega, D_\omega, \pi_\omega, \Omega_\omega)$ and via Eq. (1.6) a map Φ_ω from $\mathscr{S}(\mathbb{R}^n)$ to $\mathcal{L}(D_\omega, D_\omega)$. Hence the 4-tuple $(\mathcal{H}_\omega, D_\omega, \Phi_\omega, \Omega_\omega)$ is a quantum field, and would be a hermitian quantum field if the $\Phi_\omega(f)$ would be essentially self-adjoint for real-valued f. This is unfortunately something we can not get for free. Hence it is – at least in the present context – advantageous to drop the self-adjointness condition from Def. 1.15. In that case we get two one-to-one correspondences:

$$\text{Hermitian quantum field} \longleftrightarrow \text{Representations of } \mathcal{A}$$

and

$$\text{Sequences of Wightman distributions} \longleftrightarrow \text{States of } \mathcal{A}.$$

Let us come back to Wightman fields. The next proposition tells us how the additional properties translate into properties of the Wightman distributions.

Proposition 1.35 ([SW64], Chapter 3.3)**.** *Let $(\mathcal{H}, D, \Phi, \Omega, U)$ a Wightman quantum field. Then the following statements hold.*

(a) All the $W^{(m)}$ are Poincaré-invariant, so

$$W^{(m)}((a,\Lambda)f) = W^{(m)}(f)$$

where $((a,\Lambda)f) = f(\Lambda^{-1}(x_1 - a), \ldots)$.

(b) (Spectrum condition). The Fourier transform $\hat{W}^{(m)}$ of $W^{(m)}$ has support in the set of $p = (p_1, \ldots, p_m) \in \mathbb{R}^{n\cdot m}$ with $\sum_{j=1}^m p_j = 0$ and $\sum_{j=k}^m p_j \in V^+$ for any $k = 2, \ldots, m$.

(c) (Locality). Whenever x_k and x_{k+1} are spacelike separated, we have

$$W^{(m)}(x_1, \ldots, x_k, x_{k+1}, \ldots, x_m) = W^{(m)}(x_1, \ldots, x_{k+1}, x_k, \ldots, x_m).$$

(d) (Cluster property).

$$\lim_{\|a\|\to\infty} W^{(m)}(x_1,\dots,x_k,x_{k+1}+a,\dots,x_m+a) = W^{(k)}(x_1,\dots,x_k)W^{(m-k)}(x_{k+1},\dots,x_m)$$

Remark 1.36 (Interpretation)**.** The $W^{(m)}$ can be regarded as correlation functions. E.g. $W^{(2)}$ – which is often called the 3-point function – describes the correlations between two field operators $\Phi(f)$, $\Phi(g)$ in the vacuum. If $W^{(2)}(f\otimes g)$ factorizes, measurements of $\Phi(f)$ and $\Phi(g)$ in the vaccuum are uncorrelated. By the cluster property this happens if the supports of f and g are very far apart. Spacial separation is, on the other hand, not sufficient. We only get invariance under permutations of f and g. A closer analysis shows that the correlations of the fields between spacelike separated events in the vaccum representation can be arbitrarily high, if the distance between the events is small enough. Poincaré invariance shows that these correlations look always the same in all inertial frames. In particular translations are interesting since it allows us to rewrite $W^{(2)}$ in terms of a distribution $W_2\in\mathscr{S}(\mathbb{R}^4)$ such that we formally get

$$W^{(2)}(f\otimes g)=\int_{\mathbb{R}^4}\int_{\mathbb{R}^4} W^{(2)}(x,y)f(x)g(y)\,dx\,dy=\int_{\mathbb{R}^4}\int_{\mathbb{R}^4} W_2(x-y)f(x)g(y)\,dx\,dy.$$

Similarly we can rewrite $W^{(m)}$ in terms of a distribution W_m in $\mathscr{S}(\mathbb{R}^{(m-1)\cdot 4})$. The W_m have interesting analyticity properties which can be used to prove structural results like the PCT theorem The spectrum condition translates to support properties in momentum space.

The last result in this chapter combines the construction of a quantum field from a sequence of Wightman distributions discussed above with discussion from the last proposition. In a nutshell it says that if the $W^{(m)}$ have all the properties from Proposition 1.35, the reconstructed field is a Wightman QF.

Theorem 1.37 ([SW64], Theorem 3.7)**.** *Consider $\mathbb{R}^n=\mathbb{R}^4$. Let a state $\omega:\mathcal{A}\to\mathbb{C}$ of the BU-Algebra $\mathcal{A}$ be given and consider the corresponding GNS representation $(\mathcal{H}_\omega,D_\omega,\pi_\omega,\Omega_\omega)$. Further let $\omega(f)=\sum_{\alpha=0}^m W^{(\alpha)}(f^{(\alpha)})$ for $W^{(\alpha)}\in\mathscr{S}'(\mathbb{R}^{4\alpha})$ and let $W^{(m)}$ satisfy the four conditions from Proposition 1.35. Then there exists a strongly continous unitary representation $U_\omega:\mathcal{P}_+^\uparrow\to\mathcal{U}(\mathcal{H}_\omega)$ such that $(\mathcal{H}_\omega,D_\omega,\pi_\omega,\Omega_\omega,U_\omega)$ is a Wightman quantum field, although without the self-adjointness of the field operators.*

2 The Free Scalar Field

2.1 Representation of the Poincaré Group

Definition 2.1. *Let a group G, a $\mathbb{C}$-vector space V and $\xi : G \times G \to \mathbb{R}$ be given. A map $U : G \to \mathrm{GL}(V)$ is called projective representation of G if*

$$U(f)U(g) = e^{i\xi(f,g)}U(fg).$$

The exponents ξ can not be freely chosen, but has to satisfy some contraints, which can be easily deduced (e.g. look at $U(f)U(g)U(h)$). If all of them are satisfied, ξ is called a cocycle.

By Wigner, a free, relativistic elementary particle is described by a projective unitary representation of $\mathcal{P}_+^\uparrow$ which actually has to be irreducible emphasizing on the term *elementary*. So Wigners goal was to classify all the projective representations of $\mathcal{P}_+^\uparrow$. For more on this topic, we refer to [Bog+90, Chapter 7.2] and [RS75, Chapter IX.8]. The representations of $\mathcal{P}_+^\uparrow$ can be divided into classes via the mass, given by $m^2 \in \mathbb{R}$. Here the physical case obviously is $m^2 \geq 0$. All representations with $m^2 > 0$ are further characterized, up to unitary equivalence, by the spin $s \in \frac{1}{2}\mathbb{N}_0 = \{0, \frac{1}{2}, 1, \frac{3}{2}, \ldots\}$ of the representation. There exists a unique unitary projective representation of $\mathcal{P}_+^\uparrow$ with $m^2 > 0$, $s \in \frac{1}{2}\mathbb{N}_0$.

We will proceed with the easiest case $m^2 > 0$, $s = 0$. The first step is to define the "mass shell"

$$H_m = \{p \in \mathbb{R}^4 \,|\, \eta(p,p) = m^2, p^0 > 0\} \,. \tag{2.1}$$

Note, that H_m is an orbit of $\mathrm{SO}^\uparrow(1,3)$ since for $\Lambda \in \mathrm{SO}^\uparrow(1,3)$, $p \in \mathcal{H}_m$ we have $\Lambda p \in \mathcal{H}_m$.

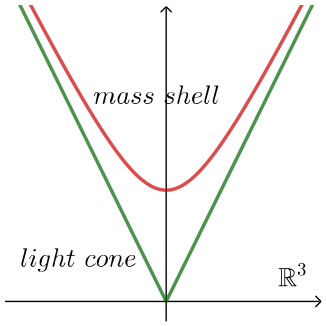

Figure 4: The mass shell from equation (2.1) as part of the forward light cone.

We parametrize H_m by

$$\mathbb{R}^3 \ni p \longmapsto j(p) = (\omega(p), p) \in H_m \tag{2.2}$$

with $\omega(p) = \sqrt{\|p\|^2 + m^2}$. This j can be regarded as a coordinate system such that H_m becomes a smooth manifold. An only slightly more detailed analysis shows that it actually is a smooth submanifold of $\mathbb{R}^4$. This shows in paticular that H_m is a locally

compact topological sapce which is homeomorphic to $\mathbb{R}^3$. Hence we can consider Borel subsets of H_m and define the following Lorentz invariant measure.

Proposition 2.2 ([RS75], Thm IX.37)**.** *For all Borel subsets $\Omega \subset H_m$ we write*

$$\Omega_m(\Omega) = \int_{j^{-1}(\Omega)} \frac{dp}{\omega(p)}.$$

This defines a Borel measure on H_m which is Lorentz invariant: $\Omega_m(\Lambda\Omega) = \Omega_m(\Omega)$ for all $\Lambda \in \mathrm{SO}^\uparrow(1,3)$ and all Borel sets $\Omega \subset H_m$.

With Ω_m we can define the Hilbert space $\mathrm{L}^2(H_m, \Omega_m)$ of square-integrable functions on H_m. Since Ω_m is Lorentz invariant, a transformation $f(\,\cdot\,) \mapsto f(\Lambda^{-1}\,\cdot\,)$ with a Lorenz transformation Λ leads to a unitary operator $U(\Lambda)$ on $\mathrm{L}^2(H_m, \Omega_m)$, and therefore to a unitary representation of $\mathrm{SO}^\uparrow(1,3)$ which is easily shown to be strongly continuous, i.e. the map

$$\mathrm{SO}^\uparrow(1,3) \ni \Lambda \mapsto U(\Lambda)\psi \in \mathrm{L}^2(H_m, \Omega_m) \tag{2.3}$$

is continuous for all $\psi \in \mathrm{L}^2(H_m, \Omega_m)$. Combining this with a representation of the translation group leads to a representation of $\mathcal{P}_+^\uparrow$.

Proposition 2.3. *The map $\mathcal{P}_+^\uparrow \ni (b, \Lambda) \mapsto U(b, \Lambda) \in \mathrm{U}\big(\mathrm{L}^2(H_m, \Omega_m)\big)$ with*

$$(U(b,\Lambda)\psi)(p) = e^{i\eta(b,p)}\phi(\Lambda^{-1}p).$$

is a strongly continuous, irreducible, unitary representation of the restricted Poincaré group.

Proof. The proof is easy and therefore left as an exercise. Note that strong coninuity can be defined as in Eq. (2.3). □

Remark 2.4 (Interpretation)**.** The representation just constructed contains (almost) everything we need for the quantum mechanics of one relativistic particle of mass m and spin 0. Due to strong continuity we can define the generators of the translations as self-adjoint operators P_j, $j = 0, \ldots, 3$

$$P_j\psi = -i\,\frac{d}{d\lambda} e^{i\eta(\lambda e_j, p)}\psi(p)\bigg|_{\lambda=0}$$

where the domains $D(P_j)$ are consisting of exactly those ψ for which the given limit exists; cf. Remark 1.18. Explicitly we get

$$(P_0\psi)(\omega(k),k) = \omega(k)\psi(\omega(k),k), \quad (P_j\psi)(\omega(k),k) = -k_j\psi(\omega(k),k)\ j = 1,2,3.$$

The P_j for $j = 1, 2, 3$ describe the three-momentum of the particle, P_0 is its Hamiltonian. With the reasoning already pointed out in Remark 1.18 the operator $P_b = \sum_j b_j P_j$ describes the Hamiltoian in the inertial frame moving with four-velocity $b \in H_1$. Hence,

we have almost everything we need for a quantum mechanical description. The only missing component are the position operators. They do not come out of the representation U directly, but they have to satisfiy a number of conditions. The corresponding analysis was carried out by Newton and Wigner in 1949. The resulting theory has serious locality problems as already mentioned in the introduction. We omit the discussion at that point, but refer the reader to original work.

We can use the parametrization map j to pull everything back to the Hilbert space $\mathrm{L}^2(\mathbb{R}^3)$. This is sometimes useful – in particular if we want to compare results with expressions known from the physics literature.

Proposition 2.5. *The map* $J : \mathrm{L}^2(H_m, \Omega_m) \to \mathrm{L}(\mathbb{R}^3)$ *given by*

$$(J\psi)(k) = \frac{(\psi \circ j)(k)}{\sqrt{\omega(k)}} = \frac{\psi(\omega(k), k)}{\sqrt{\omega(k)}}$$

is a unitary operator.

Proof. This is easily checked and therefore left as an exercise. □

2.2 The Klein-Gordon Equation

Our goal is to quantize the Klein-Gordon equation. Therefore, we have to have a closer look on the classical solutions and their relations to the discussion of the previous section. Hence, let us start with

$$(\Box + m^2)\Psi := \frac{\partial^2 \Psi}{\partial t^2} - \Delta\Psi + m^2\Psi = 0 \tag{2.4}$$

and

$$\Psi \in \mathcal{K}_{\mathbb{C}} := \{\Psi \in C^2(\mathbb{R}^4, \mathbb{C}) \,|\, \Psi_t(\,\cdot\,) := \Psi(t, \,\cdot\,) \in \mathscr{S}(\mathbb{R}^3)\}. \tag{2.5}$$

This somewhat strange function space arises from the desire to use the Fourier transform with respect to the position variable $x \in \mathbb{R}^3$. We are interested in solutions of the Cauchy problem with initial data in $\mathscr{S}(\mathbb{R}^3)$:

$$\Psi(0, x) = f(x) \quad \text{and} \quad \partial_t \Psi(0, x) = p(x) \quad \text{with} \quad f, p \in \mathscr{S}(\mathbb{R}^3).$$

Using the shortcut notation $\Psi_t(x) := \Psi(t, x)$ as in Eq. (2.5), now $\hat{\Psi}_t$ is the Fourier transform with respect to $x \in \mathbb{R}^3$. Fourier transforming all of (2.4) (in $\mathbb{R}^3$) gives

$$\frac{\partial^2 \hat{\Psi}_t(k)}{\partial t^2} + \|k\|^2 \hat{\Psi}_t(k) + m^2 \hat{\Psi}_t(k) = 0.$$

So for any k, we get an ordinary differential equation of second order in t which yields the solution

$$\hat{\Psi}_t(k) = b(k) e^{i\omega(k)t} + c(k) e^{-i\omega(k)t}$$

with $\omega(k) = \sqrt{\|k\|^2 + m^2} > 0$ for all k since $m > 0$. Using the initial conditions, we easily get

$$\hat{f}(k) = b(k) + c(k) \qquad \hat{p}(k) = i\omega(k)(b(k) - c(k))$$

and thus

$$b(k) = \frac{1}{2}\Big(\hat{f}(k) - \frac{i}{\omega(k)}\hat{p}(k)\Big) \qquad c(k) = \frac{1}{2}\Big(\hat{f}(k) + \frac{i}{\omega(k)}\hat{p}(k)\Big). \tag{2.6}$$

Proposition 2.6. *For all initial data $f, p \in \mathscr{S}(\mathbb{R}^3)$ there exists a unique solution $\Psi \in \mathcal{K}_{\mathbb{C}}$ to the Klein-Gordon equation which is given by*

$$\Psi(t,x) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} \Big(b(k)e^{i(k\cdot x + \omega(k)t)} + c(k)e^{i(k\cdot x - \omega(k)t)}\Big)\, d^3k \tag{2.7}$$

for all $t \in \mathbb{R}$. The functions $b(k), c(k)$ are derived from f, p as in (2.6) and $\omega(k) = \sqrt{\|k\|^2 + m^2}$.

Proof. We haven't shown yet that the presented solution really is in the set $\mathcal{K}_{\mathbb{C}}$. To this end first note that ω and $1/\omega$ are smooth functions. Furthermore they are polynomially bounded, and the same is true for all its powers. Hence, for all $f \in \mathscr{S}(\mathbb{R}^3)$ the products $\omega^n f$ and $\omega^{-n} f$ are Schwartz functions again. The same is true if we multiply for fixed $t \in \mathbb{R}$ with a phase factor $e^{\pm it\omega(k)}$. With this knowledge we conclude from Eq. (2.6) that b and c are in $\mathscr{S}(\mathbb{R}^3)$. Hence the expression in Eq. (2.7) is for fixed t the sum of the inverse Fourier transform of two Schwartz functions, which is again a Schwartz function in variable x for fixed t. □

Remark 2.7. For later use let us recall that we can multiply and divide Schwartz functions by ω and still get Schwartz functions. This follows since $\omega(k)$ is bounded from below by $m > 0$ (hence we avoid the singularities at zero) and polynomially bounded from above (the same is obviously true for all powers of ω). We have already used such an argument in Example 1.8 to define the product of a distribution and a polynomially bounded function.

Now we want to look for a real-valued solution Ψ, $f \in \mathscr{S}(\mathbb{R}^3, \mathbb{R})$. For the Fourier transform this means

$$\overline{\hat{f}(k)} = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} f(x) e^{ik\cdot x}\, dx,$$

so $\overline{\hat{f}(k)} = \hat{f}(-k)$ which implies $b(-k) = \overline{c(k)}$. Putting this into the solution we have by Proposition 2.6 gives us

$$\Psi(t,x) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} \overline{c}(-k) e^{i(k\cdot x + \omega(k)t)}\, dk + \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} c(k) e^{i(k\cdot x - \omega(k)t)}\, dk.$$

In the first integral, we substitute $k \to -k$, which produces no sign, so putting in $a(k) = \sqrt{2\omega(k)}c(k)$ gives the real-valued solution

$$\Psi(t,x) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} \left(\bar{a}(k) e^{-i(k\cdot x - \omega(k)t)} + a(k) e^{i(k\cdot x - \omega(k)t)} \right) \frac{dk}{\sqrt{2\omega(k)}}. \tag{2.8}$$

Obviously the argument of the integral is real as sum of something and its complex conjugate. Now let us introduce $A \in \mathrm{L}^2(H_m, \Omega_m)$ by $A = J^{-1}a$, where J is unitary from Proposition 2.5. We can express a in terms of A by

$$a(k) = (JA)(k) = \frac{A(\omega(k), k)}{\sqrt{\omega(k)}}.$$

Inserting this into Eq. (2.8) and using the abbreviation $\xi = (t, x)$, we get an integral over the mass shell.

$$\Psi(\xi) = \frac{1}{(2\pi)^{3/2}} \int_{H_m} \left(\overline{A}(\lambda) e^{i\eta(\xi,\lambda)} + A(\lambda) e^{-i\eta(\xi,\lambda)} \right) \frac{d\Omega(\lambda)}{\sqrt{2}}.$$

We summarize this discussion in the following two propositions.

Proposition 2.8. *Consider the sets $\mathcal{M} = \mathscr{S}(\mathbb{R}^3) \times \mathscr{S}(\mathbb{R}^3)$ and $\mathscr{S}(H_m) = J^{-1}\big(\mathscr{S}(\mathbb{R}^3)\big) \subset \mathrm{L}^2(H_m, \Omega_m)$. For each $(f,p) \in \mathcal{M}$ the function $A_{f,p} : H_m \to \mathbb{C}$ with*

$$A_{f,p}(\omega(k), k) = \frac{1}{\sqrt{2}} \Big(\omega(k) \hat{f}(k) + i\hat{p}(k) \Big) \tag{2.9}$$

is in $\mathscr{S}(H_m)$. The corresponding map $\mathcal{M} \ni (f,p) \mapsto A_{f,p} \in \mathscr{S}(H_m)$ is real linear and invertible. The inverse is given by $\mathscr{S}(H_m) \ni A \mapsto (f_A, p_A) \in \mathcal{M}$ with

$$f_A(x) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} \left(a(k) e^{ik\cdot x} + \bar{a}(k) e^{-ik\cdot x} \right) \frac{dk}{\sqrt{2\omega(k)}} \tag{2.10}$$

$$p_A(x) = \frac{i}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} \left(a(k) e^{ik\cdot x} - \bar{a}(k) e^{-ik\cdot x} \right) \frac{\sqrt{\omega(k)}}{\sqrt{2}} dk \tag{2.11}$$

with $a = JA$ and the unitary J from Proposition 2.5.

Proof. We have to show that $JA_{f,p}$ with $(f,p) \in \mathcal{M}$ and

$$JA_{f,p}(k) = \frac{1}{\sqrt{2}} \left(\sqrt{\omega(k)} \hat{f}(k) + \frac{i}{\sqrt{\omega(k)}} \hat{p}(k) \right)$$

is in $\mathscr{S}(\mathbb{R}^3)$. But this follows from the fact that f, p are Schwartz functions and Remark 2.7 above. Linearity of the map is obvious, and invertibility folows from the existence of an inverse. That the given map is really the inverse is left as an exercise to the reader (this can be easily done with slight modifications of the calculations above). □

Proposition 2.9. *For each* $A \in \mathscr{S}(H_m) = J^{-1}\big(\mathscr{S}(\mathbb{R}^3)\big) \subset \mathrm{L}^2(H_m, \Omega_m)$ *the expresssion*

$$\Psi = \sqrt{2}\,\mathrm{Re}\,\mathcal{F}\overline{A} \quad \textit{with} \quad (\mathcal{F}\overline{A})(\xi) = \frac{1}{(2\pi)^{3/2}} \int_{H_m} e^{i\eta(\xi,\lambda)}\,\overline{A}(\lambda) d\Omega(\lambda) \tag{2.12}$$

hence

$$\Psi(\xi) = \frac{1}{(2\pi)^{3/2}} \int_{H_m} \Big(\overline{A}(\lambda) e^{i\eta(\xi,\lambda)} + A(\lambda) e^{-i\eta(\xi,\lambda)}\Big) \frac{d\Omega(\lambda)}{\sqrt{2}}. \tag{2.13}$$

is a solution of the Klein-Gordon equation (2.4), and an element of the space $\mathcal{K} = \{f \in \mathcal{K}_{\mathbb{C}} \,|\, f = \overline{f}\}$ *with* $\mathcal{K}_{\mathbb{C}}$ *from Eq. (2.5). The corresponding initial data are given as* $(f, p) = (f_A, p_A)$ *with the expressions from Proposition 2.8. Similarly, we get the solution with initial data* $(f, p) \in \mathcal{M}$ *if* $A = A_{f,p}$ *holds.*

Proof. That $\mathcal{F}\overline{A}$ is a solution follows easily from

$$\begin{aligned}
\frac{\partial^2}{\partial t^2}\mathcal{F}\overline{A}(t,x) &= -\frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} e^{i(\omega(k)t - k\cdot x)} \omega(k)^2 \bar{a}(k) \frac{dk}{\sqrt{\omega(k)}} \\
(\Delta\mathcal{F}\overline{A})(t,x) &= \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} e^{i(\omega(k)t - k\cdot x)} \|k\|^2 \bar{a}(k) \frac{dk}{\sqrt{\omega(k)}}.
\end{aligned}$$

With $a = JA$ and the unitary J from Proposition 2.5. Hence $\mathrm{Re}\,\mathcal{F}\overline{A}$ is a real-valued solution. The initial data for the solution $\mathcal{F}\overline{A}$ are easily calculated as

$$(\mathcal{F}\overline{A})(0,x) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} e^{-ix\cdot k} \frac{\bar{a}(k)}{\sqrt{\omega(k)}}\, dk, \tag{2.14}$$

$$\Big(\frac{\partial}{\partial t}\mathcal{F}\overline{A}\Big)(0,x) = \frac{i}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} e^{-ix\cdot k} \bar{a}(k) \sqrt{\omega(k)}\, dk \tag{2.15}$$

and are obviously Schwartz functions (cf. Remark 2.7). Hence by Proposition 2.6 the solution $\mathcal{F}\overline{A}$ is in $\mathcal{K}_{\mathbb{C}}$, its real part, obviously, is in $\mathcal{K}$. The remaining statements follow immediately from Eqs. (2.14), (2.15) and Proposition 2.8. □

Remark 2.10 (Fourier transform)**.** In the last proposition we have implicitly introduced the map

$$\mathcal{F} : \mathscr{S}(H_m) \to C^\infty(\mathbb{R}^4),\ A \mapsto \mathcal{F}A \quad \text{with} \quad \mathcal{F}A(\xi) = \int_{H_m} e^{i\eta(\xi,\lambda)} g(\lambda)\, d\Omega(\lambda)$$

and the domain

$$\mathscr{S}(H_m) = J^{-1}\big(\mathscr{S}(\mathbb{R}^3)\big) \subset \mathrm{L}^2(H_m, \Omega_m).$$

Since J is unitary, $\mathscr{S}$ is an isomorphic copy of $\mathscr{S}(\mathbb{R}^3)$. Hence we can equip it with the same topology such that it becomes a Fréchet space. This implies in particular that we can define the topological dual $\mathscr{S}'(H_m)$ and embed $\mathscr{S}(H_m)$ into $\mathscr{S}'(H_m)$ via a map

$\mathscr{S}(H_m) \ni A \mapsto \phi_A \in \mathscr{S}'(H_m)$ with

$$\phi_A(f) = \int_{H_m} A(\lambda) f(\lambda)\, d\Omega_m(\lambda).$$

All our knowledge about tempered distributions can be translated to the "mass shell supported" versions in $\mathscr{S}(H_m)$ via the map J.

The map $\mathcal{F}$ can be regarded as some form of Fourier transform and as such it can be extended to the distribution space $\mathscr{S}'(H_m)$. This can be done in terms of

$$E : \mathscr{S}(\mathbb{R}^4) \ni f \mapsto \sqrt{2\pi}\tilde{f}\big|_{H_m} \quad \text{with} \quad \tilde{f}(\lambda) = \frac{1}{2\pi}\int_{\mathbb{R}^4} e^{i\eta(\xi,\lambda)} f(\xi)\, d\xi. \tag{2.16}$$

Note that we have defined the Fourier transform $\tilde{f}$ for this particular purpose with the Minkowski metric η rather than the usual scalar product. With the map E we write

$$\begin{aligned}
\phi_A(Ef) &= \sqrt{2\pi}\int_{H_m} \frac{1}{(2\pi)^2}\int_{\mathbb{R}^4} e^{i\eta(\xi,\lambda)} f(\xi)\, d\xi\, A(\lambda)\, d\Omega(\lambda) \\
&= \int_{\mathbb{R}^4} \left[\frac{1}{(2\pi)^{3/2}}\int_{H_m} e^{i\eta(\xi,\lambda)} A(\lambda)\, d\Omega_m(\lambda)\right] f(\xi)\, d\xi \\
&= \int_{\mathbb{R}^4} \mathcal{F}A(\xi) f(\xi)\, d\xi = \phi_{\mathcal{F}A}(f)
\end{aligned}$$

with $\phi_{\mathcal{F}A} \in \mathscr{S}'(\mathbb{R}^4)$ the regular distribution belonging to $\mathcal{F}A \in C^\infty(\mathbb{R}^4)$ – please check that this is well-defined although $\mathcal{F}A$ is not in L^p, cf. Example 1.6. Summarizing this calculation we can conclude that $\mathscr{S}'(H_m) \ni \phi \mapsto \phi \circ E \in \mathscr{S}'(\mathbb{R}^4)$ is the (unique) weakly contiunous extension of $\mathcal{F}$ to $\mathscr{S}'(H_m)$; cf. the discussion in Remark 1.7. The following proposition shows how to recreate the real solution $\operatorname{Re}\mathcal{F}$ in terms of the map E.

Proposition 2.11. *For each $A \in \mathscr{S}(H_m)$ the complex linear extension of the map*

$$\mathscr{S}(\mathbb{R}^4, \mathbb{R}) \ni f \mapsto \sqrt{2}\operatorname{Re}\phi_{\overline{A}}(Ef) = \sqrt{2}\operatorname{Re}\int_{H_m} Ef(\lambda)\overline{A}(\lambda) d\Omega(\lambda)$$

coincides with the regular distribution $\phi_{\sqrt{2}\operatorname{Re}\mathcal{F}\overline{A}}$

$$\phi_{\sqrt{2}\operatorname{Re}\mathcal{F}\overline{A}}(f) = \int_{\mathbb{R}^4} \sqrt{2}\operatorname{Re}\mathcal{F}\overline{A}(\xi) f(\xi) d\xi$$

associated to the solution $\sqrt{2}\operatorname{Re}\mathcal{F}\overline{A}$ of the Klein-Gordon equation.

Proof. We calculate $\phi_{\overline{A}}(Ef)$ with real-valued test function f:

$$\begin{aligned}\sqrt{2}\,\mathrm{Re}\,\phi_{\overline{A}}(Ef) &= \frac{1}{\sqrt{2}}\left(\phi_{\overline{A}}(Ef) + \overline{\phi_{\overline{A}}(Ef)}\right)\\ &= \int_{H_m} \overline{A}(\lambda)\frac{1}{(2\pi)^{3/2}}\int_{\mathbb{R}^4} e^{i\eta(\lambda,\xi)} f(\xi)\,d\xi\,d\Omega(\lambda) + \\ &\qquad \int_{H_m} A(\lambda)\frac{1}{(2\pi)^{3/2}}\int_{\mathbb{R}^4} e^{-i\eta(\lambda,\xi)} f(\xi)\,d\xi\,d\Omega(\lambda)\end{aligned}$$

Hence

$$\sqrt{2}\,\mathrm{Re}\,\phi_{\overline{A}}(Ef) = \int_{\mathbb{R}^4} f(\xi)\sqrt{2}\,\mathrm{Re}(\mathcal{F}\overline{A})(\xi)\,d\xi,$$

as claimed. □

It is left as an Exercise to the reader that we can use this strategy to generate weak (i.e. distributional) solutions of the Klein-Gordon equation. For us it is important to remember this proposition when we introduce the free quantum field in the next section.

Proposition 2.12. *Consider $(b,\Lambda)\in\mathcal{P}_+^\uparrow$ and the action of $\mathcal{P}_+^\uparrow$ on functions $f:\mathbb{R}^4\to\mathbb{C}$ given by $f\mapsto(b,\Lambda)f$ with $[(b,\Lambda)f](\xi) = f\big(\Lambda^{-1}(\xi-b)\big)$. The map $\mathscr{S}(H_m)\ni A\mapsto \mathrm{Re}\,\mathcal{F}\overline{A}$ intertwines this action with the representation $U(b,\Lambda)$ introduced in Proposition 2.3. In other words*

$$(b,\Lambda)\,\mathrm{Re}\,\mathcal{F}\overline{A} = \mathrm{Re}\,\mathcal{F}\overline{U(b,\Lambda)A}$$

holds for all $(b,\Lambda)\in\mathcal{P}_+^\uparrow$ and all $A\in\mathcal{S}(H_m)$.

Proof. We write $\Psi(\xi) = \mathrm{Re}(\mathcal{F}\overline{A})(\xi)$ and calculate

$$\begin{aligned}\Psi(\Lambda^{-1}(\xi-b)) &= \frac{1}{(2\pi)^{3/2}}\int_{H_m}\left(\overline{A(\lambda)}e^{i\eta(\Lambda^{-1}(\xi-b),\lambda)} + A(\lambda)e^{-i\eta(\Lambda^{-1}(\xi-b),\lambda)}\right)d\Omega_m(\lambda)\\ &= \frac{1}{(2\pi)^{3/2}}\int_{H_m}\left(\overline{A(\lambda)}e^{i\eta(\xi-b,\Lambda\lambda)} + A(\lambda)e^{-i\eta(\xi-b,\Lambda\lambda)}\right)d\Omega_m(\lambda).\end{aligned}$$

By substituting $\Lambda\lambda\to\tilde{\lambda}$, due to Lorentz invariance of the measure Ω_m we get

$$\begin{aligned}\Psi(\Lambda^{-1}(\xi-b)) &= \frac{1}{(2\pi)^{3/2}}\int_{H_m}\left(\overline{A(\Lambda^{-1}\tilde{\lambda})}e^{i\eta(\xi-b,\tilde{\lambda})} + A(\Lambda^{-1}\tilde{\lambda})e^{-i\eta(\xi-b,\tilde{\lambda})}\right)d\Omega_m(\tilde{\lambda})\\ &= \frac{1}{(2\pi)^{3/2}}\int_{H_m}\Big(e^{i\eta(x,\tilde{\lambda})}\underbrace{\overline{e^{i\eta(b,\tilde{\lambda})}A(\Lambda^{-1}\tilde{\lambda})}}_{=\overline{(U(b,\Lambda)A)(\tilde{\lambda})}} + \dots\Big)\,d\Omega_m(\tilde{\lambda})\\ &= \mathrm{Re}\left(\mathcal{F}\overline{U(b,\Lambda)A}\right)(\xi)\end{aligned}$$

which concludes the proof. □

Remark 2.13 (Canonical formalism)**.** We can regard the space $\mathcal{M} = \mathscr{S}(\mathbb{R},\mathbb{R})\times\mathscr{S}(\mathbb{R},\mathbb{R})$ of initial data as the classical phase space of the system. The classical Hamilton function

is given by

$$\mathcal{M} \ni (f,p) \mapsto \mathfrak{h}(f,p) = \frac{1}{2}\langle p,p\rangle + \frac{1}{2}\langle (m^2-\Delta)f, f\rangle \in \mathbb{R}$$

where $\langle\,\cdot\,,\,\cdot\,\rangle$ denotes the *real* standard scalar product on $\mathscr{S}(\mathbb{R}^3,\mathbb{R})$, i.e. $\langle f,g\rangle = \int_{\mathbb{R}^3} f(x)g(x)dx$. Using Gateaux differentials we can easily define partial derivatives of $\mathfrak{H}$ as

$$\frac{\partial}{\partial f}\mathfrak{h}(f,p) \in \mathscr{S}(\mathbb{R}^3) \quad \text{with} \quad \left\langle \frac{\partial}{\partial f}\mathfrak{h}(f,p), v\right\rangle = \frac{d}{d\lambda}\mathfrak{h}(f+\lambda v,p)\big|_{\lambda=0}$$
$$\frac{\partial}{\partial p}\mathfrak{h}(f,p) \in \mathscr{S}(\mathbb{R}^3) \quad \text{with} \quad \left\langle \frac{\partial}{\partial p}\mathfrak{h}(f,p), w\right\rangle = \frac{d}{d\lambda}\mathfrak{h}(f,p+\lambda w)\big|_{\lambda=0}$$

with $v,w \in \mathscr{S}(\mathbb{R}^3)$. It is easy to see that these partial differentials exist in the given sense and have the values

$$\frac{\partial}{\partial f}\mathfrak{h}(f,p) = (\Delta^2 - m^2)f \quad \text{with} \quad \frac{\partial}{\partial p}\mathfrak{h}(f,p) = p.$$

Now consider a curve $\mathbb{R} \ni t \mapsto (f_t,p_t) \in \mathcal{M}$ which is differentiable in the sense that $(t,x) \mapsto f_t(x)$ and $(t,x) \mapsto p_t(x)$ are C^1-functions on $\mathbb{R}^4$. Then we can define the time derivative $(\dot{f}_t,\dot{p}_t) \in \mathcal{M}$ in terms of partial derivatives

$$\dot{f}_t(x) = \frac{\partial}{\partial t}f_t(x), \quad \dot{p}_t(x) = \frac{\partial}{\partial t}p_t(x),$$

such that the Hamilton equations become

$$\dot{f}_t = \frac{\partial}{\partial p}\mathfrak{h}(f_t,p_t) = p_t \qquad \dot{p}_t = -\frac{\partial}{\partial f}\mathfrak{h}(f_t,p_t) = (m^2-\Delta^2)f_t. \tag{2.17}$$

Differentiating a second time and inserting the second equation into the first shows that $(t,x) \mapsto f(t,x)$ has to satisfy the Klein-Gordon equation. By Proposition 2.9 we can therefore conclude that the system of equations (2.17) has a unique solution for all initial data $f_0 = f$, $p_0 = p$. We just take the unique solution $\psi \in \mathcal{K}$ of the Klein-Gordon equation with initial data f,p and define $f_t(x) = \psi(t,x)$ and $p_t(x) = \partial_t\psi(t,x)$. Hence Hamiltons equation with the Hamilton function $\mathfrak{h}$ represent a reinterpretation of the Cauchy problem for the Klein-Gordon equation as an infinite-dimensional Hamiltonian system; i.e. something like infinite-dimensional classical mechanics.

We are discussing this topic since we want to justify the interpretation of $\mathcal{M}$ as the classical phase space of our system, and f,p as the canonical variables. With substantially more effort we could also introduce (in a mathematically rigorous way) Poisson brackets and see that f,p satisfy some kind of "canonical" Poisson relations (maybe I will add this in a future version of these notes). This reinterpretation in a "canonical" (i.e. Hamiltonian) way is useful in the context of quantization. A possible strategy to quantize the field system we replace the classical fields $f(x)$ and $p(x)$ by operator-valued fields $\varphi(x),\pi(x)$ satisfying a version of "canonical commutation relations", and generate the Hamiltonian H of the quantum theory from $\mathfrak{h}$ by replacing f and p with φ and π. The spacetime field

$\Phi(t,x)$ then is generated by evolving $\varphi(x)$ with time evolution $\exp(itH)$.

While this procedure basically works (although with a substantial reinterpretation, which is nesscessary in order to make the math work rigorously) this is not the most adequate way to proceed. At that point the mass shell function $A_{f,p} \in \mathscr{S}(H_m) \subset \mathrm{L}^2(H_m, \Omega_m)$ comes into play. Since the map $(f,p) \mapsto A_{f,p}$ is invertible, we can regard the space $\mathscr{S}(H_m)$ as an alternative version of the phase space, which is parametrized in terms of complex variables $A, \overline{A}$ rather then f, p. Everything we have expressed in terms of f, p can be reexpressed with $A, \overline{A}$. E.g. the Hamilton function $\mathfrak{h}$ can be written as a function of $a = JA$ and $\bar{a}$ as follows.

$$\mathfrak{h}(a,\bar{a}) = \frac{1}{2}\int_{\mathbb{R}^3} \omega(k)\big(\bar{a}(k)a(k) + a(k)\bar{a}(k)\big)dk \tag{2.18}$$

It is left as an exercise to the reader to check this equation and to translate it into an integral over H_m involving $A, \overline{A}$ rather than $a, \bar{a}$.

The advantage of the variables $A, \overline{A}$ over f, p is the Poincaré covariance. The definition of the initial data f, p requires a split of spacetime into space and time, and this fixes an inertial frame up to spatial rotations and translations. Hence, there is not one canonical formalism, but there is (roughly speaking) a different one for each inertial frame. To understand why this is different for the mass shell functions $A, \overline{A}$, consider two inertial frames Σ_1, Σ_2 and a Poincaré transformation (b, Λ) transforming from Σ_1 into Σ_2. If a field is described in Σ_1 by a spacetime function ψ it is described in Σ_2 by $(b,\Lambda)\psi$. Hence if ψ is given by $A \in \mathscr{S}$ via $\psi = \sqrt{2}\,\mathrm{Re}\,\mathcal{F}\overline{A}$ the transformed field is given by $U(b,\Lambda)A$; cf. Proposition 2.12. Therefore the space $\mathscr{S}(H_m)$ does not only contain one but *all* canonical descriptions for all inertial frames and the unitaries $U(b,\Lambda)$ represent the transformation from ψ to $(b,\Lambda)\psi$. This intrinsic covariance is a great advantage if we are aiming at a quantum theory which has a similar covariance – like a Wightman quantum field theory.

Our strategy to quantize the Klein-Gordon equation is therefore to replace the functions $A, \overline{A}$ in all expressions developed so far by appropriately chosen operator-valued fields (satisfying some commutation relations which are motivated by the Poisson relations I have skipped). Applying this strategy in particular to the expression in (2.13) leads to an operator-valued spacetime field which (with an appropriate mathematical reinterpretation of the steps just outlined) will become our Wightman field.

2.3 The Free Quantum Field

First we need to make some general remarks on Fock spaces[12]. Let $\mathcal{H}$ be a separable Hilbert space. The belonging Fock space then is given by

$$\mathcal{F}(\mathcal{H}) = \mathbb{C} \oplus \Big(\bigoplus_{n=1}^{\infty} \mathcal{H}^{\otimes n}\Big).$$

[12] For more on the topic of Fock spaces of relativistic particles, we refer to [Bog+90, Chapter 7.3]

Now for $\sigma \in S_n$, where S_n is the permutation group on n, we define

$$V_\sigma : \mathcal{H}^{\otimes n} \to \mathcal{H}^{\otimes n} \qquad \sigma(\Psi_1 \otimes \ldots \otimes \Psi_n) = \Psi_{\sigma^{-1}(1)} \otimes \ldots \otimes \Psi_{\sigma^{-1}(n)}$$

and based on this, the symmetrization operator

$$S_n : \mathcal{H}^{\otimes n} \to \mathcal{H}_+^{\otimes n} \subset \mathcal{H}^{\otimes n} \qquad S_n = \frac{1}{n!} \sum_{\sigma \in S_n} V_\sigma.$$

Note, that S_n is a projection. Now the range $\mathcal{H}_+^{\otimes n}$ of S_n is called symmetric or Bose subspace. With this, we further define the symmetric Fock space as follows.

$$\mathcal{F}_+(\mathcal{H}) = \mathbb{C} \oplus \Big(\bigoplus_{n=1}^{\infty} \mathcal{H}_+^{\otimes n} \Big)$$

With this reminder, we can return to the free quantum field. We orient ourselves towards [RS75, Chapter X.7]. For some $f \in \mathcal{H}$ we define

$$b^-(f) : \mathcal{H}^{\otimes n} \to \mathcal{H}^{\otimes (n-1)} \qquad \Psi_1 \otimes \ldots \otimes \Psi_n \longmapsto \langle f, \Psi_1 \rangle \Psi_2 \otimes \ldots \otimes \Psi_n$$

A short calculation shows $\|b^-(f)\| = \|f\|$ so b^- is bounded and it can be linearly extended to all of $\mathcal{H}^{\otimes n}$ for all $n > 0$. For $n = 0$ we define $b^-(f) = 0$. It is easy to check that $b^+(f) = (b^-(f))^*$ on product vectors acts like

$$b^+(f) = (\Psi_1 \otimes \ldots \otimes \Psi_n) = f \otimes \Psi_1 \otimes \ldots \otimes \Psi_n.$$

With this, we want to define creation and annihilation operators as already known from the quantum harmonic oscillator.

For a densely defined self-adjoint operator $H : D(H) \to \mathcal{H}$ on $\mathcal{H}$ we explain

$$F_0 = \{\Psi \in \mathcal{F}_+(\mathcal{H}) \,|\, \exists_{n \in \mathbb{N}} \forall_{m > n} \Psi^{(m)} = 0\}$$
$$\tilde{D}_H = \{\Psi \in F_0 \,|\, \Psi^{(n)} \in D(H) \otimes \ldots \otimes D(H) \ \forall_n\}$$

where F_0 is the set of finite particle vectors. For $\Psi \in \tilde{D}_H$ we define

$$(d\Gamma(H)\Psi)^{(n)} = (H \otimes \mathbb{1} \otimes \ldots \otimes \mathbb{1} + \mathbb{1} \otimes H \otimes \mathbb{1} \otimes \ldots \otimes \mathbb{1} + \ldots)\Psi^{(n)}.$$

This is called second quantization. A similar construction can be done for unitaries $U : \mathcal{H} \to \mathcal{H}$ where

$$(\Gamma(U)\Psi)^{(n)} = (U \otimes \ldots \otimes U)\Psi^{(n)}$$

is said second quantization with

$$\Gamma(e^{itH}) = e^{it\, d\Gamma(H)}.$$

Now in order to define creation and annihilation operators, we need the number operator which is given by $N = d\Gamma(\mathbb{1})$. For $\Psi \in \mathcal{H}_+^{\otimes n}$ we obviously have $N\Psi = n\Psi$. Note that N is essentially self-adjoint on F_0. Sadly, N is a global observable whereas the lab setting obviously is local so on its own it is not as useful. This is why we define the annihilation operator on $\mathcal{F}_+(\mathcal{H})$ with domain F_0 to be

$$A(f) = \sqrt{N + \mathbb{1}}b^-(f).$$

For each $\Psi, \Phi \in F_0$ we have

$$\langle \Psi, A(f)\Phi\rangle = \langle \Psi, \sqrt{N+\mathbb{1}}b^-(f)\Phi\rangle = \langle \sqrt{N+\mathbb{1}}\Psi, b^-(f)\Phi\rangle = \langle S(b^-(f))^*\sqrt{N+\mathbb{1}}\Psi, \Phi\rangle$$

which implies that creation operator is given by

$$A^*(f) = (A(f))^* = S(b^-(f))^*\sqrt{N+\mathbb{1}} = \sqrt{N}Sb^+(f).$$

It is worth noting that both $A(f)$ and $A^*(f)|_{F_0}$ are closable. We denote their closures by $A(f)$ and $A^*(f)$.

Example 2.14. Consider a topological space with Borel measure (M, μ) and the associated Hilbert space $\mathcal{H} = L^2(M, \mu)$. By [RS80, Chapter II.4], we then have

$$\mathcal{H}^{\otimes n} = L^2(\underbrace{M \times \ldots \times M}_{n \text{ arguments}}, \mu \otimes \ldots \otimes \mu) \qquad \mathcal{H}_+^{\otimes n} = L^2_S(M \times \ldots \times M, \mu \otimes \ldots \otimes \mu)$$

where L^2_S is the set of symmetric functions[13]. The operators A and A^* are given by

$$(A(f)\Psi)^{(n)}(m_1, \ldots, m_n) = \sqrt{n+1}\int_M \overline{f(m)}\Psi^{(n+1)}(m, m_1, \ldots, m_n)\, d\mu(m)$$

$$(A^*(f)\Psi)^{(n)}(m_1, \ldots, m_n) = \frac{1}{\sqrt{n}}\sum_{j=1}^{n} f(m_j) \cdot \Psi^{(n-1)}(m_1, \ldots, \underset{\text{skip}}{\hat{m}_j}, \ldots, m_n).$$

Definition 2.15. *The Segal quantization over $\mathcal{H}$ on F_0 is defined via*

$$\mathcal{H} \ni f \longmapsto \Phi_S(f) = \frac{1}{\sqrt{2}}(A(f) + A^*(f)) \in \mathcal{F}_+(\mathcal{H}).$$

Note that the Segal quantization is $\mathbb{R}$-, but not $\mathbb{C}$-linear since $f \mapsto b^-(f)$ is an antilinear map.

Theorem 2.16 ([RS75], Theorem X.41)**.** *The following statements hold.*

(a) (Self-adjointness). For all $f \in \mathcal{H}$, $\Phi_S(f)$ is essentially self-adjoint on F_0.

[13] This means that L^2_S is the set of functions in L^2 which are invariant under permutations of the coordinates.

(b) (Cyclicity of the vacuum). For the vacuum $\Omega = 1 \oplus 0 \oplus 0 \oplus \dots$

$$\operatorname{span}\{\Phi_S(f_1)\cdots\Phi_S(f_n)\Omega \,|\, f_1,\dots,f_n \in \mathcal{H}, n \in \mathbb{N}\} \subset \mathcal{F}_+(\mathcal{H})$$

is dense.

(c) (Commutation relations). For each $\Psi \in F_0$, $f, g \in \mathcal{H}$ *we have*

$$[\Phi_S(f), \Phi_S(g)]\Psi = i \operatorname{Im}\langle f, g\rangle_{\mathcal{H}}\Psi.$$

Further, for the unitary operator $w(f) = \exp(i\Phi_s(f))$ *(Weyl operator) we have*

$$W(f+g) = e^{-i\operatorname{Im}\langle f,g\rangle}W(f)W(g).$$

(d) (Continuity). If $f_n \to f$ *in* $\mathcal{H}$, *then*

$$\begin{aligned} W(f_n)\Psi &\to W(f)\Psi \quad \text{for all } \Psi \in \mathcal{F}_S(\mathcal{H}) \\ \Phi_S(f_n)\Psi &\to \Phi_s(f)\Psi \quad \text{for all } \Psi \in F_0 \end{aligned}$$

(e) (Covariance condition). For all unitary operators U *on* $\mathcal{H}$,

$$\Gamma(U) : D(\overline{\Phi_S(f)}) \to D(\overline{\Phi_S(Uf)})$$

and for $\Psi \in D(\overline{\Phi_S(Uf)})$ *we have*

$$\Gamma(U)(\overline{\Phi_S(f)})\Gamma(U)^*\Psi = \overline{\Phi_S(Uf)}\Psi$$

for all $f \in \mathcal{H}$.

We can now use the Segal quantization to define the free Hermitian scalar field of mass m. To this end note that we can write $\Phi_S = \sqrt{2}\operatorname{Re}A^*$ which is (almost) the same expression we have already used in Proposition 2.11 to rewrite solutions to the Klein-Gordon equation in a distributional sense. The only difference is the quantity A which is now an operator rather than a function (and complex conjugation becomes taking adjoint). Hence with the map E from Equation (2.16), which was given by

$$E : \mathscr{S}(\mathbb{R}^4) \to L^2(H_m) \qquad f \longmapsto \sqrt{2\pi}\,\tilde{f}|_{H_m}$$

using the 4-dimensional Fourier transform

$$\tilde{f}(k) = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^4} f(x)e^{i\eta(x,k)}\,dx,$$

the free Hermitian scalar field of mass m now is defined to be

$$\Phi_m(f) = \Phi_S(Ef) = \sqrt{2}\operatorname{Re}A^*(Ef)$$

on $f \in \mathscr{S}(\mathbb{R}^4, \mathbb{R})$ and

$$\Phi_m(f) = \Phi_S(\mathrm{Re}(f)) + i\Phi_S(\mathrm{Im}(f))$$

for arbitrary $f \in \mathscr{S}(\mathbb{R}^4)$. Applying the reasoning from Proposition 2.11 we see that *formally* this $\Phi_m(f)$ is an operator-valued solution of the Klein-Gordon equation. Exact reasoning leads to the following theorem.

Theorem 2.17 ([RS75], Theorem X.42)**.** *The 5-tuple*

$$\Big(\mathcal{F}_+\big(L^2(H_m)\big), F_0, \Phi_m, \Omega, \Gamma\big(U(\cdot,\cdot)\big)\Big)$$

is a Wightman quantum field where

$$\big(U(b,\Lambda)\Psi\big)(p) = e^{i\eta(\Lambda,b)}\Psi(\Lambda^{-1}p)$$

is the unitary representation of the restricted Poincaré group on $L^2(H_m, \Omega_m)$. Further

$$\Phi_m\big((\Box + m^2)f\big) = 0$$

holds for each $f \in \mathscr{S}(\mathbb{R}^4)$.

Proof. Most of the statement can be derived easily from earlier results, like the discussion of the Klein-Gordon equation in Sec. 2.2 or properties of the Segal quantization in Theorem 2.16. Only a few properties (like self-adjointness) require more work. In those cases we refer the reader to the corresponding proof in [RS75]. □

Remark 2.18 (Free Hamiltonian)**.** Following the reasoning from Remark 1.18 we can define the free Hamiltonian of the theory (in the inertial system in which we are at rest) as the generator of the time translations, i.e. as the self-adjoint operator H_0 satisfying

$$\Gamma\big(U(te_0, \mathbb{1})\big) = \exp(itH_0), \quad \forall t \in \mathbb{R}.$$

Using the properties of second quantization $\Gamma(U)$ this can be rewritten as

$$H_0 = d\Gamma(P_0) \quad \text{with} \quad (P\psi)(\omega(k), k) = \omega(k)\psi(k, \omega(k)),$$

where $\psi \in \mathrm{L}^2(H_m, \omega_m)$ has to be chosen such that $P\psi$ is square-integrable again. This gives the domain of P_0. We come back to H_0 in Remark 2.24, where we derive a different expression for it which is more familiar from the physics literature.

In Sec. 2.2 we have used the freedom to rewrite functions on the mass shell as functions on $\mathbb{R}^3$ by using the unitary operator $J : \mathrm{L}^2(H_m, \Omega_m) \to \mathrm{L}^2(\mathbb{R}^3)$ from Proposition 2.5. We can do the same with the field operators by applying the second quantization of J to $\Phi_m(f)$. For real-valued f this leads to

$$\tilde{\Phi}_m(f) = \Gamma(J)\Phi_m(f)\Gamma(J)^* = \frac{1}{\sqrt{2}}\left(a(Ef) + a^*(Ef)\right)$$

with the creation and annihilation operators a^*, a on $\mathcal{F}_+\big(\mathrm{L}^2(\mathbb{R}^3)\big)$. We have used lower case letters here, in order to make it easier to distinguish them from the corresponding operators on $\mathcal{F}_+\big(\mathrm{L}^2(H_m,\Omega)\big)$. The relations to the latter are

$$\Gamma(J)A(f)\Gamma(J)^* = a\Big(\frac{f\circ j}{\sqrt{\omega}}\Big) \qquad \Gamma(J)(A(f))^*\Gamma(J)^* = a^*\Big(\frac{f\circ j}{\sqrt{\omega}}\Big).$$

Note that these are operators on $\mathcal{F}_+\big(\mathrm{L}^2(\mathbb{R}^3)\big)$. As before the finite particle vectors $F_0 \subset \mathcal{F}_+\big(\mathrm{L}^2(\mathbb{R}^3)\big)$ form the domain of these operators. In slight abuse of notation we here have kept the same symbol (F_0) we have already used for the finite particle vectors in $\mathcal{F}_+\big(\mathrm{L}^2(H_m,\Omega_m)\big)$, although strictly speaking both sets are different.

We now turn to the question whether we can evaluate $\tilde{\Phi}_m$ at spacetime events rather than test functions. The answer, basically, is yes, but the result of such an evaluation is not an operator. To explain the details we need some preparations. (Also note that we only treat the case $\tilde{\Phi}_m$. The mass shell based field Φ_m can be treated similarly. To work out the corresponding details is left as an exercise to the reader).

On the domain

$$D_{\mathscr{S}} = \{\Psi \in F_0 \,|\, \Psi^{(n)} \in \mathscr{S}((\mathbb{R}^3)^n) \text{ for all } n\}$$

we can define the annihilation operator a_k at momentum k via

$$(a(k)\Psi)^{(n)}(k_1,\ldots,k_n) = \sqrt{n+1}\,\Psi^{(n+1)}(k,k_1,\ldots,k_n), \tag{2.19}$$

which is a well defined operator and related to $a(f)$ with test function $f \in \mathrm{L}^2(\mathbb{R}^3)$ by the following Proposition.

Proposition 2.19. *For all $f \in \mathrm{L}^2(\mathbb{R}^3)$ the annihilation operator $a(f)$ can be written as*

$$a(f) = \int_{\mathbb{R}^3} \overline{f(k)} a_k \, dk$$

with a_k from Eq. (2.19).

Proof. This is straightforward and again left as an exercise. □

Basically, we would like to do something similar, but if we try to calculate the adjoint a_k^* of a_k *formally* we get

$$(a^*(k)\Psi)^{(n)}(k_1,\ldots,k_n) = \frac{1}{\sqrt{n}}\sum_{j=1}^{n}\delta(k-k_j)\Psi^{(n-1)}(k_1,\ldots,\hat{k}_j\ldots,k_n) \tag{2.20}$$

with the obvious problem that δ occurs, so $a^*(\cdot)$ is not a densely defined operator. We can rectify this by using quadratic forms

Remark 2.20 (Quadratic forms. [RS80], Sec. VIII.6)**.** A quadratic form q on a Hilbert space $\mathcal{H}$ is a map $q : Q(q) \times Q(q) \to \mathbb{C}$, where $Q(q) \subset \mathcal{H}$ is a dense, linear subspace (the

form domain), and q is linear in the second and conjugate linear in the first argument. If X is an operator with dense domain $D(X)$ we can immediately define a quadratic form $X[\phi,\psi] = \langle\phi, X\psi\rangle$ with domain $D(X)$. The converse is not true. There are quadratic forms which do not belong to an operator (we will see an example very soon). Hence, quadratic forms are more singular objects than operators. Nevertheless the notation $\langle\psi, q\phi\rangle$ is frequently used for $q[\psi,\phi]$ even if q is not an operator. In the following we will use the phrase "X and q coincide in the sense of quadratic forms" for an operator X and a quadratic form q if $q[\psi,\phi] = \langle\psi, X\phi\rangle$ holds for all ψ,ϕ in an appropriate domain.

Now assume a_k^* would exist as an operator. Then we could assign it to the quadratic form

$$a_k^*[\psi,\phi] = \langle\psi, a_k^*\phi\rangle = \langle a_k\psi,\phi\rangle \quad \psi,\phi \in D_{\mathscr{S}}$$

Hence, although a_k^* is not an operator the quantity on the right hand side of this equation only uses a_k and is therefore well defined. We use this expression as the definition of a_k^* *as a quadratic form*. Now in analogy to Proposition 2.19 we get the following.

Proposition 2.21. *For all $f \in \mathrm{L}^2(\mathbb{R}^3)$ we have*

$$a^*(f) = \int_{\mathbb{R}^3} f(k) a_k^*\, dk$$

in the sense of quadratic forms.

Proof. Again, this is straightforward. □

Now, we can apply this to the field $\tilde{\Phi}_m$ and get the following result.

Theorem 2.22. *The quantity*

$$\Phi_m(t,x) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} \left(e^{i(\omega(k)t - kx)} a_k^* + e^{-i(\omega(k)t-kx)} a_k \right) \frac{d^3k}{\sqrt{2\omega(k)}} \tag{2.21}$$

is a well defined quadratic form on $\mathcal{F}_+\big(\mathrm{L}^2(\mathbb{R}^2)\big)$ with domain $D_{\mathscr{S}}$. For a test function $f \in \mathscr{S}(\mathbb{R}^4)$ it is related to the field $\tilde{\Phi}_m(f)$ by

$$\tilde{\Phi}_m(f) = \int_{\mathbb{R}^4} f(x,t)\Phi_m(x,t)dxdt.$$

Proof. The fact that $\Phi_m(t,x)$ is a well defined quadratic form is obvious from the definitions. The relation to $\Phi_m(f)$ can be cheked with the arguments already used in the proof of Proposition 2.11. [In a future version of these notes I might expand this proof.] □

Remark 2.23 (Wick ordering)**.** For this remark, cf. [Haa96, Chapter I.5.2]. The idea to use quadratic forms in order to evaluate quantities at spacetime events can be extended to powers of the field like $\Phi^n(t,x)$ or more generally expressions involving polynomials

in the a_k, a_k^*. If we look in particular at $(a_k^*)^n(a_k)^m$ we can easily use the same idea as above and just "move all a_k^* to the left under the scalar product":

$$\langle \phi, (a_k*)^n(a_k)^m\psi\rangle = \langle (a_k)^n\phi, (a_k)^m\psi\rangle =: (a_k^*)^n(a_k)^m[\phi,\psi]. \tag{2.22}$$

If we change the ordering of the a_k, a_k^*, however this procedure fails. In the product $a_k a_k^*$ we can not move a_k^* "to the left" since a_k and a_k^* are not commuting. Hence we have to move a_k first which would produce an (undefined) operator a_k^* in the left argument of the scalar product. To understand the problem we are facing let us see what happens if we calculate the vaccum expectation value $\langle \Omega, a_k a_k^*\Omega\rangle$ of $a_k a_k^*$ with the formal expression from Eq. (2.20).

$$\langle \phi, a_k a_k^*\psi\rangle = \langle a_k^*\Omega, a_k^*\Omega\rangle = \int_{\mathbb{R}^3} \delta^2(k-p)dp.$$

The integral on the right hand side involves the square of the delta function which can not be defined within distribution theory. Hence, the given vacuum expectation value is just infinite. Usually, physical quantities like energy should have vacuum expectation value zero (naively speaking, the vacuum should just mean no physical particles). In that sense the infinities we see are just artifacts arising from wrong operator ordering. To get the real physical quantities we just have to subtract these infinities, and this can be done by choosing the correct operator ordering. The only ordering where no problems in terms of inifinities arise is the one in Eq. (2.22). Hence the simple rule is: Whenever we encounter a monomial in a_k, a_k^* is: "move all creation operators to the left". This is known as Wick or normal ordering and usually indicated by colons written to the left and the right of an expression. E.g. to calculate $:\Phi(t,x)^n:$ we formally expand $\Phi(t,x)^n$ into a polynomial of a_k, a_k^* and in any monomial we get that way, we move all a_k^* to the left.

Remark 2.24 (Operator ordering and quantization)**.** Operator ordering problems as the ones just described are not uncommon in quantum theory and already happen within the quantization of one non-relativistic particle. The (too) simple rule which tries to map classical observables (functions on phase space) to quantum observables (operators on a Hilbert space) is to replace the canonical phase space coordinates q_j, p_k by position and momentum operators Q_j, P_k and to replace each occurence of q_j, p_k in a phase space function F by these operators. But even if F is just a polynomial in q_j, p_k this procedure is ambiguous since the q_k, p_k mutually commute such that $q_j p_j = p_j q_j$ while $Q_j P_j$ and $P_j Q_j$ are different operators. Hence, we need additional (physical) arguments in order to make the operator ordering unambiguous. In the given example we might want to use $\frac{1}{2}(Q_j P_j + P_j Q_j)$ since this combination is at least hermitian.

Based on these considerations we can describe the basic quantization rule for scalar fields as follows:

1. Rewrite a classical field observable (i.e. a function $F : \mathcal{M} \to \mathbb{R}$ on classical phase space $\mathcal{M}$; cf. the disicussion in Remark 2.13) as a polynomial in the complex valued phase space variables $a(k), \bar{a}(k)$.

2. Replace these functions by the creation and annihilation operators (or more precisely quadratic forms) $\tilde{a}_k, \tilde{a}_k^*$. (Note that we are adding a tilde to the operators here in order to make the distinction from the functions $a, \bar{a}$ easier. We will drop this tilde after this remark is finished).
3. Apply Wick ordering, in order to get rid of operator ordering ambiguities and the corresponding infinities. The choice of $\tilde{a}_k$, $\tilde{a}_k^*$ as the proper replacement of $a(k), \bar{a}(k)$ is motivated by a comparison of the commutation relations between $\tilde{a}_k$, $\tilde{a}_k^*$ on the one hand and the Poisson relations between $a(k)$ and $\bar{a}(k)$ on the other.

This receipt provides a clear rule to turn any polynomial in $a(k)$, $\bar{a}(k)$ into a quadratic form on the Hilbert space $\mathcal{F}_+\big(\mathrm{L}^2(\mathbb{R}^3)\big)$. If we apply it to the classical solution of the Klein-Gordon Equation from (2.8) we get the quantum field $\Phi(t,x)$ as written in Eq. (2.21). Another possible application is the classical Hamilton function $\mathfrak{h}(a,\bar{a})$ from Eq. (2.18).

$$:\mathfrak{h}(\tilde{a}_k, \tilde{a}_k^*): \;= \int_{\mathbb{R}^3} \omega(p) : \underbrace{\frac{1}{2}(\tilde{a}_k^* \tilde{a}_k + \tilde{a}_k \tilde{a}_k^*)}_{\text{harmonic oscillator}} : dp = \int_{\mathbb{R}^3} \omega(k) \tilde{a}_k^* \tilde{a}_k \, dp = H_0$$

Please check yourself that (as a quadratic form) this really coincides with the free Hamiltonian H_0 from Remark 2.18. Also note in this context that the number operator N can be given by a similar expression as

$$N = \int_{\mathbb{R}^3} \tilde{a}_p^* \tilde{a}_p \, dp.$$

Finally note that exactly the same discussion can be given in terms of the mass shell functions $A(\lambda), \overline{A}(\lambda)$ and the corresponding quadratic forms $\tilde{A}_\lambda$, $\tilde{A}_\lambda$ on $\mathcal{F}_+\big(\mathrm{L}^2(H_m, \Omega_m)\big)$. Working this out is left as an exercise.

For the last point in this section we have to talk about are the Wightman distributions of $\Phi_m(t,x)$. They are calculated in the next theorem.

Theorem 2.25. *For $m, n \in \mathbb{N}_0$ with $m > 1$ the Wightman distributions of the free field Φ_m from Theorem 2.17 are given by $W^{(2n+1)} = 0$ and*

$$W^{(2m)}(f_1 \otimes \ldots \otimes f_{2m}) = \sum_{\sigma \in pair} W^{(2)}(f_{\sigma(1)} \otimes f_{\sigma(2)}) \ldots W^{(2)}(f_{\sigma(2n-1)} \otimes f_{\sigma(2m)}) \quad (2.23)$$

where $pair \subset S_{2n}$ is the set of permutations which satisfy $\sigma(1) < \sigma(3) < \ldots < \sigma_{(2n-1)}$ and $\sigma_{(2k+1)} < \sigma_{(2k+2)}$ for $k > 0$. The two-point function $W^{(2)}$ is given by

$$W^{(2)}(f \otimes g) = \frac{1}{2(2\pi)^3} \int_{H_m} \int_{\mathbb{R}^4} \int_{\mathbb{R}^4} e^{i\eta(y-x,p)} f(x) g(y) \, dx \, dy \, d\Omega_m(p),$$

which can formally be rewritten as

$$W^{(2)}(f \otimes g) = \frac{1}{i} \int_{\mathbb{R}^4} \int_{\mathbb{R}^4} \Delta_+(y-x; m^2) f(x) g(y)\, dx\, dy$$

where

$$\Delta_+(x; m^2) = \frac{i}{2(2\pi)^3} \int e^{i\eta(x,p)} \frac{d^3p}{\omega(p)}.$$

Proof. The combinatorial structure of the $W^{(n)}$ given here is a consequence of the structure of the Segal quantization and therefore not special to the free field. A corresponding proof can be given using Wick's theorem. Typically, this is left as an exercise. In lack of a good reference I am following this tradition. Please check validity of (2.23) yourself. To calculate the two-point function we use $Ef = \sqrt{2\pi}\tilde{f}$ and further get

$$\begin{aligned}
W^{(2)}(f \otimes g) &= \langle \Omega_m, \Phi_m(f)\Phi_m(g)\Omega_m \rangle = \langle \Phi_m(\overline{f})\Omega_m, \Phi_m(g)\Omega_m \rangle \\
&= \frac{2\pi}{2} \langle a^*(\overline{\tilde{f}}), a^*(\hat{g}) \rangle = \frac{2\pi}{2} \langle \overline{\tilde{f}}, g \rangle_{L^2(H_m)} \\
&= \frac{2\pi}{2} \int_{H_m} \frac{1}{(2\pi)^2} \int_{\mathbb{R}^4} e^{-i\eta(x,p)} f(x)\, dx \, \frac{1}{(2\pi)^2} \int_{\mathbb{R}^4} e^{i\eta(y,p)} g(y)\, dy\, d\Omega_m(p) \\
&= \frac{1}{2(2\pi)^3} \int_{H_m} \int_{\mathbb{R}^4} \int_{\mathbb{R}^4} e^{i\eta(y-x,p)} f(x) g(y)\, dx\, dy\, d\Omega_m(p)
\end{aligned}$$

Written formally, this yields

$$W^{(2)}(f \otimes g) = \frac{1}{i} \int_{\mathbb{R}^4} \int_{\mathbb{R}^4} \Delta_+(y-x; m^2) f(x) g(y)\, dx\, dy$$

where

$$\Delta_+(x; m^2) = \frac{i}{2(2\pi)^3} \int e^{i\eta(x,p)} \frac{d^3p}{\omega(p)},$$

as claimed. □

Fields with this structure of the n-point functions are called generalized free fields or quasi-free. The Wightman axioms imply that $W^{(2)}$ has to be of the following form

$$W^{(2)}(f \otimes g) = \int_{\mathbb{R}^4} \int_{\mathbb{R}^4} \int_0^\infty \Delta_+(x-y; m^2)\, d\rho(m)\, f(x) g(y)\, dx\, dy$$

where $d\rho(m)$ is a polynomial bounded measure. This is also called the Källen-Lehmann-representation, see [RS75, Theorem IX.34].

2.4 Time-Zero Fields

Before we can define the time-zero field, we need the following short result.

Proposition 2.26. *For $f \in \mathscr{S}(\mathbb{R}^3)$ we define the dual space element $\delta f \in \mathscr{S}'(\mathbb{R}^4)$ via*

$$(\delta f)(g) := \int_{\mathbb{R}^4} \delta(t) g(t,x) f(x)\, dt\, dx = \int_{\mathbb{R}^3} g(0,x) f(x)\, dx.$$

Then the map E from (2.16) can be extended to distributions of the form δf.

Proof. Using the definition of δf we get

$$\begin{aligned}(E(\delta f))(\omega(k),k) &= \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^4} \delta(t) f(x) e^{i\omega(k)t} e^{-ik\cdot x}\, dt\, dx \\ &= \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} f(x) e^{-ik\cdot x}\, dx = \hat{f}(k),\end{aligned}$$

so $E(\delta f)(j(k)) = \hat{f}(k)$ with j from (2.2). □

Definition 2.27. *We define the time-zero fields to be*

$$\varphi_m(f) = \Phi_S(E(\delta f)) \qquad \pi_m(f) = \Phi_S(i\omega E(\delta f))$$

for $f \in \mathscr{S}(\mathbb{R}^3, \mathbb{R})$ and its $\mathbb{C}$-linear extension onto all of $\mathscr{S}(\mathbb{R}^3)$.

As in the spacetime field Φ_m we can use the unitary $\Gamma(j)$ to transform φ_m and π_m into operators on $\mathcal{F}_+\big(\mathrm{L}^2(\mathbb{R}^3)\big)$. We define

$$\tilde{\varphi}_m(f) = \Gamma(J)\varphi_m(f)\Gamma(J)^* \quad \text{and} \quad \tilde{\pi}_m(f) = \Gamma(J)\pi_m(f)\Gamma(J)^*.$$

In analogy to Theorem 2.22 we can express $\tilde{\varphi}_m(f)$ and $\tilde{\pi}_m(f)$ as "smeared out versions" of appropriately chosen quadratic forms.

Proposition 2.28. *The time-zero fields $\tilde{\varphi}_m$, $\tilde{\pi}_m$ can be written in terms of a_k and a_k^* as quadratic forms via*

$$\tilde{\varphi}_m(f) = \int f(x)\varphi_m(x)\, dx$$

where

$$\varphi_m(x) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \left(e^{-ik\cdot x} a_k^* + e^{ik\cdot x} a_k\right) \frac{dk}{\sqrt{2\omega(k)}}$$

and

$$\tilde{\pi}_m(g) = \int_{\mathbb{R}^3} g(x)\pi_m(x)\, dx$$

with

$$\pi_m(x) = \frac{i}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} \omega(k) \Big(e^{-ik\cdot x} a_k^* - e^{ik\cdot x} a_k \Big) \sqrt{\frac{\omega(k)}{2}}\, d^3k.$$

Proof. In the case of $\tilde{\varphi}_m$ this follows immediately from Theorem 2.22 by setting $t = 0$. In the case of $\tilde{\pi}_m$ we use (2.21) and take a time derivative to get

$$\partial_t \Phi_m(t,x) = \frac{i}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} \Big(\omega(k) e^{i(\omega(k)t - kx)} a_k^* - \omega(k) e^{-i(\omega(k)t - kx)} a_k \Big) \frac{d^3k}{\sqrt{2\omega(k)}}$$

and thus

$$(\partial_t \tilde{\Phi}_m)(f) = \tilde{\Phi}_m(i\omega f) = \Gamma(J) \Phi_S(\omega E(if)) \Gamma(J)^+.$$

Hence, setting $t = 0$ again leads to the statement about $\tilde{\pi}_m$. □

Remark 2.29 (Time evolution)**.** We can look at the fields $\varphi_m(x)$ and $\pi_m(x)$ as local observables which we can evolve in time (i.e. we are in the Heisenberg picture). With the free Hamiltonian H_0 from Remark 2.18 we can define

$$e^{-itH_0} \varphi_m(f) e^{itH_0} = \varphi_{m,t}(f) \qquad e^{-itH_0} \pi_m(g) e^{itH_0} = \pi_{m,t}(g). \tag{2.24}$$

This again yields

$$\begin{aligned} e^{-itH_0} \varphi_m(x) e^{itH_0} &= \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \Big(e^{-ik\cdot x} \underbrace{e^{-itH_0} a_k^* e^{itH_0}}_{= e^{it\omega(k)} a_k^*} + e^{ik\cdot x} e^{-itH_0} a_k e^{itH_0} \Big) \frac{dk}{\sqrt{2\omega(k)}} \\ &= \Phi_m(t,x). \end{aligned}$$

In other words we can reconstruct the spacetime field $\Phi_m(x,t)$ from the time-zero field and time evolution, which in turn is part of the representation $\Gamma\big(U(b,\Lambda)\big)$ of the Poincaré group. This observation leads to the question whether we can turn this into a strategy to costruct new models from a set of time-zero fields and a representation of $\mathcal{P}_+^\uparrow$, or at least a Hamiltonian. The latter depends on physical input like the type of interactions we want to describe. For the fields, however, we'd like to have a universal construction, which is completely independent from any dynamical input. We might even hope that the time-zero fields developed for the free field are already sufficient. To explain up to which degree this idea works or does not work is the main task of the rest of this section.

Remark 2.30 (Canonical commutation relations)**.** To understand the last remark a bit better, let us have a short look at the quantum mechanical description of N non-relativistic particles. The Hilbert space of this system is $\mathrm{L}^2(\mathbb{R}^n)$ with $n = 3N$ and the most important observables are position and momentum described by self adjoint operators

Q_j, P_k which are defined for $\psi \in \mathscr{S}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$ by

$$\big(Q_j\psi\big)(x) = x_j\psi(x), \quad \big(P_j\psi\big)(x) = \frac{1}{i}\frac{\partial\psi}{\partial x_j}(x). \tag{2.25}$$

The Q_j and P_k should be reagarded as a the finite degree of freedom analog of the time-zero fields $\varphi_m(x)$, $\pi_m(x)$. The most important structural property of the Q_j, P_k are the canonical commutation relations, which (in their most elementary form) are given as

$$[Q_jQ_k]\psi = [P_j, P_k]\psi = 0, \qquad [Q_j, P_k]\psi = i\delta_{jk}\psi \tag{2.26}$$

for any $\psi \in \mathscr{S}(\mathbb{R}^n)$. These relations are the main reason why the Q_j and P_k are chosen in the given way: We are following Dirac's quantization rule which says: "Quantization means to replace Poisson brackets by operator commutators". The Q_j and P_k should be the quantum analogs of position and momentum coordinates q_j, p_k of the classical phase space $\mathbb{R}^n \times \mathbb{R}^n$ and the relations in (2.26) exactly resemble the Poisson relations between q_j and p_k. Large parts of the physics literature even tell the legend that up to unitary equivalence the Q_j, P_k from Eq. (2.25) are the only possible choice for operators satisfying the commutation relations in (2.26). If we assume for the moment that this is true (we will come to that in a minute), and if we accept that the CCRs in (2.26) are mandatory requirements, we can conclude that the operators in (2.25) are the only possible choice – completely independent from the dynamics our model should obey. The latter is completely contained in the Hamiltonian of the system.

What can we learn from this for field theory? First of all note that the expressions for $\varphi_m(x)$ and $\pi_m(x)$ in Proposition 2.28 resemble Eqs. (2.10) where we have expressed the intial data $(f, p) \in \mathcal{M}$ for the classical solution in terms of the complex variables $a, \bar{a}$. According to the discussion in Remark 2.24 the fields $\varphi_m(x)$ and $\pi_m(x)$ are the quantizations of the canonical phase space variables $f(x)$ and $p(x)$. Since the latter are the field thoeretic replacement of phase space coordinates q_j, p_k (cf. our discussion in Remark 2.13) this observation justifies the claim that $\varphi_m(x)$ and $\pi_m(x)$ are the field theoretic analogs of the operators Q_j, P_k. Hence let us have a look at the commutation relations they satisfy:

$$[\tilde{\varphi}_{m,t}(f), \tilde{\varphi}_{m,t}(g)]\psi = [\tilde{\pi}_{m,t}(f), \tilde{\pi}_{m,t}(g)]\psi = 0 \qquad [\tilde{\varphi}_{m,t}(f), \tilde{\pi}_{m,t}(g)] = i\langle f, g\rangle, \tag{2.27}$$

where on the right hand side of the second equation we have used the scalar product in $\mathrm{L}^2(\mathbb{R}^3)$, and $\psi \in F_0$ is chosen arbitrarily. Rewriting this in a formal way we get the more often used form

$$[\varphi_{m,t}(x), \varphi_{m,t}(y)] = [\pi_{m,t}(x), \pi_{m,t}(y)] = 0 \qquad [\varphi_{m,t}(x), \pi_{m,t}(y)] = i\delta(x - y). \tag{2.28}$$

The latter version can be regarded as the natural "continuous variable version" of the relations in (2.26). Therefore we proceed in analogy to the finite degree of freedom case as follows. We declare the CCRs in Eq. (2.28), or maybe better the mathematically more rigorous version in (2.27), as the fundamental quantization condition each choice of time-

zero fields (for scalar field theories) should satisfy. If there would be again a uniqueness result, the time-zero fields constructed for the free field would be the only choice and we could proceed along the lines outlined in Remark 2.29. The big amounts of subjunctives in the last sentence already indicate that there is a problem with this procedure to explain why we have to clarify first, in which sense the Q_j and P_k are really unique. This step is prepared by the following definition.

Definition 2.31. *Consider a locally convex, real vector space V which also carries a real scalar product $\langle \cdot, \cdot \rangle$, and strongly continuous maps $f \mapsto U(f)$, $g \mapsto V(g)$ from V into the unitary group $\mathcal{U}(\mathcal{H})$ of a separable Hilbert space $\mathcal{H}$. If for all $f_1, f_2 \in V$ the operators U, V satisfy the Weyl relations*

$$\begin{aligned} V(f_1 + f_2) &= V(f_1)V(f_2) \\ U(f_1 + f_2) &= U(f_1)U(f_2) \\ V(f)U(g) &= U(g)V(f)\exp(\langle f, g\rangle), \end{aligned}$$

then $\{U, V\}$ is called a representation of the Weyl relations over V.

Example 2.32 (Schrödinger representation)**.** The central reference for all statements concerning the Schrödinger representation is the book of Folland [Fol16]. Consider $V = \mathbb{R}^n$ and the operators Q_j, P_k from Eq. (2.25). Real linear combinations of the Q_j and of the P_k are self-adjoint operators (on appropriate domains containing $\mathscr{S}(\mathbb{R}^n)$ as a subspace). Hence with $f, g \in \mathbb{R}^n$ we can define

$$U(f) = \exp\left(i\sum_{j=1}^{n} f_j Q_j\right), \quad \text{and} \quad V(g) = \exp\left(i\sum_{k=1}^{n} g_k P_k\right). \tag{2.29}$$

By Stone's theorem [RS80, Sec. VIII.4] these maps are strongly continuous. It is also easy to check that

$$\big(U(f)\psi\big)(x) = \exp\left(\sum_{j=1}^{n} x_j f_j\right)\psi(x), \quad \text{and} \quad \big(V(g)\psi\big)(x) = \psi(x+g)$$

hold. From the last equation it is easy to see that the $U(f)$, $V(g)$ satisfy the Weyl relations. Hence the maps $\mathbb{R}^n \ni f \mapsto U(f)$ and $\mathbb{R}^n \ni g \mapsto V(g)$ form a representation of the Weyl relation, which is called the *Schrödinger representation*. A deeper analysis also shows that this representation is irreducible, i.e. the only bounded operators commuting with all $U(f)$ and $V(g)$ are multiples of the identity.

Theorem 2.33 (Stone-von Neumann)**.** *The Schrödinger representation is the only irreducible representation of the Weyl relations over $\mathbb{R}^n$ up to unitary equivalence.*

Proof. See [RS80, Thm VII.I.14]. □

Remark 2.34 (Canonical commutation relations)**.** Assume self-adjoint operators Q_j, P_k are given such that

1. we can define $U(f)$, $V(g)$ as in Eq. (2.29),
2. these $U(f)$, $V(f)$ are representation of the Weyl relations and
3. they are irreducible, then we can conclude according to the Stone-von Neumann theorem that these Q_j, P_k are unitarily equivalent to the choice in Eq. (2.25).

In that case the relations in (2.26) are an easy consequence of the Weyl relations. We will rephrase this situation in the following by saying the Q_j, P_k satisfy the *Weyl form of the CCR*. If, however, we only know that the Q_j, P_k are self-adjoint and satisfy the relations in (2.26) we *can not* deduce the Weyl relations. All corresponding calculations you might find are formal and can not be made rigorous *without additional assumptions*. In general, (2.26) is not even suffcient to gurantee that $U(f)$ and $V(g)$ can be defined according to (2.29). As a consequence, there actually are examples for self-adjoint operators Q, P satisfying (2.26), which are *not* unitarily equivalent to the version in (2.25); cf. [RS80, Sec. VIII.5].

With this remark we can return to the time-zero fields. We define a corresponding representation of the Weyl relations – this time over $V = \mathscr{S}(\mathbb{R}^3)$ rather than $V = \mathbb{R}^n$ and ask for uniqueness. The depressing answer is given in the next theorem.

Theorem 2.35. *Let φ_m, π_m be the time-zero field and conjugate momentum of the free scalar field of mass m. Then*

$$U_m(f) = \exp(i\varphi_m(f)) \qquad V_m(g) = \exp(i\pi_m(g))$$

is an irreducible representation of the Weyl relations over $\mathscr{S}(\mathbb{R}^4, \mathbb{R})$. Further, (U_{m_1}, V_{m_1}) and (U_{m_2}, V_{m_2}) are unitarily inequivalent if the masses are different, so if $m_1 \neq m_2$.

Proof. [RS75, Thm X.46]. □

Remark 2.36 (Interpretation)**.** The theorem shows that we actually can consider the Weyl form of the CCR as a mandatory requirement for time-zero fields, but this does not lead to a unique choice. Even worse, the example for inequivalent representations given depend on the mass. But mass is a dynamical parameter since it labels different versions of the field equations. Hence the representations of the Weyl realtions we have to deal with in field theory *does contain dynamical information*.

Remark 2.37 (Time evolution again)**.** The failure of unitary equivalence has another striking consequence concerning time evolution. To explain what we've just lost, let us have another look at the finite degree of freedom case. Here we can look at the observables Q_j, P_k in the Heisenberg picture and time dependent operators

$$Q_j(t) = \exp(-itH)Q_k \exp(itH), \quad P_k(t) = \exp(-itH)P_k \exp(itH),$$

where H is a self-adjoint operator and the Hamiltonian of the model. If we assume the other way round that time dependent, self-adjoint operators $Q_j(t), P_k(t)$ are given which satisfy the Weyl form of the CCR at each instance of time, we can conclude from the Stone-von Neumann theorem that there is a unitary $T(t)$ satisfying $T(t)Q_j(0)T(t)^* =$

$Q_j(t)$ and $T(t)P_k(0)T(t)^* = P_k(t)$. It is still unclear (without further knowledge) whether these $U(t)$ are given as $\exp(-itH)$ in terms of a Hamiltonian H, but at least we know that the time evolution is given by unitaries on the same Hilbert space.

The lack of uniqueness in the field theoretical case indicates therefore that we can not expect that time evolution is unitary. This leads to the urgent question whether we can preserve uniqueness of the Weyl relations at least on a more abstract level. The rest of this section is devoted to an answer of this question. As a preparation we need some material about C*-algebras.

Definition 2.38. *Consider the space $\mathcal{B}(\mathcal{H})$ of bounded operators on a (not necessarily separable) Hilbert space $\mathcal{H}$. A linear subspace $\mathcal{A}$ which is closed under products[14] and adjoints[15] is called a C*-algebra.*

Definition 2.39. *An invertible linear map $\alpha : \mathcal{A}_1 \to \mathcal{A}_2$ between two C*-algebras $\mathcal{A}_1$, $\mathcal{A}_2$ is called *-isomorphism if $\alpha(AB) = \alpha(A)\alpha(B)$ and $\alpha(A^*) = \alpha(A)^*$ holds for all $A, B \in \mathcal{A}_1$. If $\mathcal{A}_1 = \mathcal{A}_2 = \mathcal{A}$, the map α is called *-automorphism.*

Remark 2.40 (C*-algebras)**.** The central references for all statements about C*-algebras are the books of Bratteli and Robinson [BR12; BR02]. The following is a short list of additional remarks.

1. C*-algebras can be defined alternatively in an abstract way. We have chosen the explicit form as algebras of operators since it leads us more directly to the desired goal.
2. C*-algebras are in particular *-algebras. Hence all the corresponding material from Sec. 1.3 applies. This in particular concerns states and representations.
3. Consider the GNS representation $(\mathcal{H}_\omega, D_\omega, \pi_\omega, \Omega_\omega)$ of a state ω of a C*-algebra $\mathcal{A}$.
4. In the definition of *-isomorphisms we haven't added any boundedness requirements, because this is not necessary. All *-isomorphisms α satisfy $\|\alpha(A)\| = \|A\|$ for all $A \in \mathcal{A}_1$. Hence they are automatically bounded ($\|\cdot\|$ here denotes the operator norm).

Definition 2.41. *Consider a real pre-Hilbert space $(\mathcal{K}, \langle\cdot,\cdot\rangle)$ which is also a locally convex space, and a representation $\{U, V\}$ of the Weyl relations. The smallest C*-algebra $\mathcal{A}$ containing the operators $U(f), V(g)$ for all $f, g \in \mathcal{K}$ is called the Weyl-algebra of $\{U, V\}$.*

Theorem 2.42. *The Weyl algebras $\mathcal{A}_1$, $\mathcal{A}_2$ of two representations $\{U_1, V_1\}$ and $\{U_2, V_2\}$ over the same space $(\mathcal{K}, \langle\cdot,\cdot\rangle)$ are *-isomorphic. More precisely, there is a *-isomorphism $\alpha : \mathcal{A}_1 \to \mathcal{A}_2$ with $\alpha\big(U_1(f)\big) = U_2(f)$ and $\alpha\big(V_1(g)\big) = V_2(g)$ for all $f, g \in \mathcal{K}$.*

Proof. [BR02, Theorem 5.2.8] □

[14] $A, B \in \mathcal{A} \Rightarrow AB \in \mathcal{A}$
[15] $A \in \mathcal{A} \Rightarrow A^* \in \mathcal{A}$

Remark 2.43 (Interpretation)**.** Assume that by some construction we got time-zero fields φ_t and π_t which by all instances of time satisfy the Weyl form of the CCR, then by the previous theorem we get a family of *-isomorphisms $\alpha(t)$ with $\alpha_t\big(\exp(i\varphi_0(f))\big) = \exp(i\varphi_t(f))$ and similarly for π_t. These α_t take the role of the unitaries $T(t)$ from Remark 2.37, and therefore we can hope for a dynamical description of field theories in terms of automorphisms of C*-algebras, rather than unitaries on Hilbert spaces. Or to formulate this in another way: The Hilbert space structure which served us well in quantum mechanics is too rigid for field theory and has to be replaced by C*-algebras. This point of view is quite successful at least on the conceptual side; cf. the book of Haag [Haa96] for an in-depth discussion of this point of view. For constructive purposes, however, our problems are not completely solved. It might (and does) in particular happen that time-zero fields do not exist at all, since their existence is not a consequence of the Wightman axioms. For the most non-trivial example of an interacting model – quartic self interactions in 1+1 dimensions – the material developed so far is sufficient. We consider this case in the next chapter.

3 Interactive Fields

3.1 Naive Approach

For the first part of this chapter, we orient ourselves towards [RS75, p.233 ff.]. An "interacting field theory" is a field theory satisfying the Wightman axioms which has a nontrivial scattering theory. A natural way to construct such fields (cf. the discussion in the last section; in particular Remark 2.29) is to try to make a pertubation of some free theory, so

$$\varphi_t(x) = e^{-itH}\varphi(x)e^{itH} \qquad \pi_t(x) = e^{-itH}\pi(x)e^{itH}$$

for some Hamiltonian H. This works if H is self-adjoint but otherwise most likely breaks the Poincaré invariance.

In classical Lagrangian field theory the simplest Hamiltonians are of the form

$$H = H_0 + \underbrace{\lambda \int_{\mathbb{R}^3} F(\varphi(x))\, dx}_{=H_I}$$

where F is some function, say a polynomial with some coupling constant λ. Since we want the Hamiltonian to be bounded below, we expect that the polynomial is of even order with positive highest coefficient. The most simple non-trivial case then is $F(x) = x^4$, so

$$H = H_0 + \lambda \int_{\mathbb{R}^3} \varphi(x)^4\, dx.$$

This expression does not make sense unless we consider the Wick-ordered case *(see Chapter 2.3)*

$$H = H_0 + \lambda \int_{\mathbb{R}^3} :\varphi(x)^4:\, dx,$$

which is at least a quadratic form. Unfortunately, this quadratic form does not arise from an operator since formally computing $H\Omega$ "$=$" Ψ for $\Psi^{(n)} = 0$ if $n \neq 4$ gives

$$\Psi^{(4)}(k_1, \dots, k_n) = \int_{\mathbb{R}^3} \frac{\exp\Big(-ix\sum_{i=1}^4 k_i\Big)}{\prod_{i=1}^4 (2\pi)^{3/2}(2\omega(k_i))^{1/2}}\, dx = \frac{\delta\Big(\sum_{i=1}^4 k_i\Big)}{(2\pi)^{9/2}\prod_{i=1}^4 (2\omega(k_i))^{1/2}}$$

which certainly is not in L_2, alone because of the δ-function. This we can try to fix by smearing with a test function, so

$$\int_{\mathbb{R}^3} \frac{g(x)\exp\Big(-ix\sum_{i=1}^4 k_i\Big)}{\prod_{i=1}^4 (2\pi)^{3/2}(2\omega(k_i))^{1/2}}\, dx = \frac{\hat{g}\Big(\sum_{i=1}^4 k_i\Big)}{(2\pi)^{9/2}\prod_{i=1}^4 (2\omega(k))^{1/2}},$$

but we still do not get an L^2 function because $\omega(k_i)$ grows too slowly at infinity.

3.2 The Cutoff Hamiltonian

In order to obtain an operator, we restrict ourselves to one space dimension. The discussion of Wightman quantum fields and the construction of the free field has to be adopted accordingly. This is, however, staightforward and (again) left as an exercise for the reader. We will try to sketch the idea of the construction of the $(\varphi^4)_2$ model from [GJ68; GJ70a; GJ70b] and accompanying papers.

We replace the quadratic form $\int_{\mathbb{R}} :\varphi(x)^4: dx$ by $\int_{\mathbb{R}} g(x) :\varphi(x)^4: dx$ where $g(x)$ is a real-valued function in $L^2(\mathbb{R})$. So we have

$$H_g = H_0 + H_I(g) = \int_{\mathbb{R}} \omega(k) a_k^* a_k \, dk + \int_{\mathbb{R}} g(x) :\varphi(x)^4: dx - E_g$$

with domain

$$D(H_g) = D_{\mathscr{S}} = \{\Psi \in F_0 \,|\, \Psi^{(m)} \in \mathscr{S}(\mathbb{R}^{n \cdot m})\}.$$

Here the constant E_g, also called self energy of the vacuum, is chosen so that

$$0 = \inf\{\text{spectrum}\, H_g\}.$$

Further E_g is finite because of the spatial cutoff and because of the limitation to only one space dimension.

For the sake of locality, for g one often chooses some smooth function of compact support which equals one on a very large interval. Either way, the effect of g is to turn off the interaction for large values of x. Therefore, g is called the space cut-off and $H(g)$ is called the spatially cut-off Hamiltonian for the $(\varphi^4)_2$ field theory.

Theorem 3.1 ([RS75], Theorem X.62)**.** *The spatially cut-off Hamiltonian H_g is essentially self-adjoint for any $g \in \mathscr{S}(\mathbb{R}, \mathbb{R})$.*

Now the solution to the cutoff field equation

$$\frac{\partial^2 \Phi}{\partial t^2} - \frac{\partial^2 \Phi}{\partial x^2} + m^2 \Phi + 4\lambda g \Phi^3 = 0, \tag{3.1}$$

also compare (2.4), is given by the field at time t

$$\varphi_{t,g}(x) = \exp(-itH_g)\varphi(x)\exp(itH_g)$$

with corresponding momentum at time t

$$\pi_{t,g}(x) = \exp(-itH_g)\pi(x)\exp(itH_g).$$

Here $\varphi(x)$ and $\pi(x)$ are chosen to be the time-zero fields of the free field. Please keep this in mind, since this choice will cause considerable trouble below. Since translations affect the cutoff and therefore the field, this does not belong to a Wightman field because it violates Lorentz invariance. The simplest idea would be to take the limit $g \to 1$.

Unfortunately, this means $E_g \to \infty$ so our Hamiltonian does not converge in a proper sense.

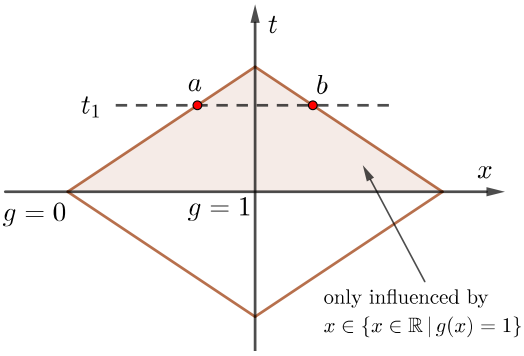

Figure 5: The field $\varphi_{t,g}(x)$ in two-dimensional space time. If $g(y) = 1$ for $|y| \leq |x| + |t|$, then the field is independent of g. This means for a time t_1, we get an associated interval (a, b) such that $\varphi_{t,g}(x) = \varphi_t(x)$ on the subset of the rhombus $(a, b) \times (-t_1, t_1)$.

However $\varphi_{t,g}(x)$ is independent of g provided $g(y) = 1$ for $|y| \leq |x| + |t|$ as is illustrated in Figure 5. This is a consequence of the fact that the field equation (3.1) propagates disturbances only with the speed of light (here set to 1), and therefore any changes to g made "far outside" can not influence anything in the rhombical region shown in the figure. Using this idea we can prove thr following

Theorem 3.2. *Consider $t_1 \in \mathbb{R}^+$ and $f_1, f_2 \in \mathscr{S}(\mathbb{R})$, with support in a bounded interval (a, b). Furthermore assume that $g \in \mathscr{S}(\mathbb{R}, \mathbb{R})$ satisfies $g(x) = 1$ for all $x \in (a-t_1, b+t_1)$. If we choose the time-zero fields $\varphi(f_1)$ and $\pi(f_2)$ of the free field we can define the quantities*

$$\begin{aligned} \varphi_t(f_1) &= \exp(-itH_g)\varphi(f_1)\exp(itH_g) \\ \pi_t(f_2) &= \exp(-itH_g)\pi(f_2)\exp(itH_g) \end{aligned} \tag{3.2}$$

for all $t \in [-t_1, t_1]$. They are self-adjoint operators (on an appropriate domain) and are independent of the cutoff g, provided it satisfies the given condition.

Proof. [GJ68, Sec. V] together with Lemma 3.2.2 and Theorem 3.2.6 of [GJ70a]. □

Hence, by choosing the interval, on which $g(x) = 1$ holds, large enough, we can define $\varphi_t(f_1)$ and $\pi_t(f_2)$ for all t and all $f_1, f_2 \in \mathscr{D}(\mathbb{R})$[16]. These fields have the cutoff removed.

From here on we can introduce spacetime fields (i.e. fields which are smeared over space and time rather than over space at a fixed time) by first defining

$$\Phi_g(f) = \int_{\mathbb{R}} \varphi_{t,g}(f_t)dt = \int_{\mathbb{R}} f(t,x)\varphi_g(t,x)dxdt \quad \text{with} \quad f \in \mathscr{D}(\mathbb{R}^2),\ f_t(\,\cdot\,) = f(t,\,\cdot\,),$$

then proving that this leads to a closable operator admitting an invariant dense domain [GJ70a, Theorem 3.2.3]. Using Theorem 3.2 we can show that this operator is independent of g provided the support of f is contained in the rhombical region from Figure 5; cf. [GJ70a, Thm 3.2.6]. In that way we get an operator-valued distribution on the Hilbert

[16] $\mathscr{D}(\mathbb{R}) = \{f \in C^\infty(\mathbb{R}) \,|\, \operatorname{supp} f \text{ compact}\}$

space $\mathcal{F}_+\big(\mathrm{L}^2(\mathbb{R})\big)$ with an invariant dense domain we haven't specified (cf. [GJ70a, Sect 3.2] for details; cf also [GJ70a, Thm 3.2.7] for the continuity requirements a distribution has to satisfy).

$$\mathscr{D}(\mathbb{R}^2) \ni f \mapsto \Phi(f) = \int_{\mathbb{R}^2} f(t,x)\varphi(t,x)dxdt, \tag{3.3}$$

which finally can be extended to test functions from $\mathscr{S}(\mathbb{R}^2)$, i.e. it is a *tempered* operator-valued distribution [GJ70a, Sec. 3.4]. We can recover the time-zero fields from Theorem 3.2 by restricting Φ and Φ_t to test "functions" δf; i.e.

$$\phi_t(f_1) = \int_{\mathbb{R}} f_1(x)\Phi(t,x)dx \quad \text{and} \quad \pi_t(f_2) = \int_{\mathbb{R}} f_2(x)\partial_t\Phi(t,x)dx.$$

For the second equation cf. [GJ70a, Theorem 3.2.5]. Note that this shows in particular that the free field and the field Φ just constructed share the time-zero fields at $t = 0$. Furthermore it can be shown that this $\Phi(f)$ satisfies the correct field equations (Theorem 3.2.5 and the following remark of [GJ70a]), is self-adjoint if f is a real-valued function [GJ70a, Sec. 3.3], and is covariant with respect to spacetime translations [GJ70a, Sec. 3.6]. Furthermore, two operators $\Phi(f_1)$ and $\Phi(f_2)$ commute if the supports of f_1 and f_2 are spacelike separated [GJ70a, Sec. 3.5]. Hence we have almost constructed a Wightman field. The most important property we haven't shown yet (apart from covariance with respect to Lorentz boosts) is the existence of a vacuum, and actually, a vacuum vector $\Omega \in \mathcal{F}_+\big(\mathrm{L}^2(\mathbb{R})\big)$ *does not exist.*

3.3 Haag's Theorem

The fact that the above approach did not work out is not surprising but this was predicted by Haag's theorem which we will state now.

Theorem 3.3 ([Bog+90], Theorem 9.28)**.** *Let Φ, Φ_0 be two scalar Wightman fields acting on Hilbert spaces $\mathcal{H}, \mathcal{H}_0$. Assume for some instance of time t we have time-zero fields*

$$\Phi(t,x), \partial_t\Phi(t,x) \qquad \Phi_0(t,x), \partial\Phi_0(t,x)$$

which are irreducible and there exists unitary T such that

$$\begin{aligned} T\Phi(t,x)T^* &= \Phi_0(t,x) \\ T\partial_t\Phi(t,x)T^* &= \partial_t\Phi_0(t,x). \end{aligned} \tag{3.4}$$

Then all Wightman functions up to $n = 4$ are identical. If Φ_0 is a free field, then Φ is also free field in the sense that all Wightman functions are identical.

We can choose for Φ the field constructed in (3.3) and the free field for Φ_0. The method outlined in the last section used the time-zero fields for the free field to construct the interacting field Φ. Hence we have $\Phi(0,x) = \Phi_0(0,x)$ and $\partial\Phi(0,x) = \partial\Phi_0(0,x)$; cf. the definition of φ_t and π_t in Eq. (3.2) and check that $\pi_t(x) = \partial_t\Phi(t,x)$ really holds (in this context also cf. [GJ70a, Theorem 3.2.5]). Hence the condition in (3.4) is satisfied for $t = 0$

and $T = \mathbb{1}$. Since Φ_0 is the free field and Φ is not, the latter can not be a Wightman field. We are seeing here consequences of the non-uniqueness of representations of the Weyl relations, as discussed in Sec. 3.3.

3.4 Remove the Cutoff

The previous section clearly tells us that we have to come up with another construction to remove the cut-off. Although there is no vacuum vector for the field φ_t from (3.4), there is a vacuum vector Ω_g in $\mathcal{F}_+\big(\mathrm{L}^2(\mathbb{R})\big)$ for H_g:

Theorem 3.4. *For each cut-off g there is (up to scalar multiples) a unique vector $\Omega_g \in \mathcal{F}_+\big(\mathrm{L}^2(\mathbb{R}^3)\big)$ which is normalized ($\|\Omega_g\| = 1$) and satisfies $H_g\Omega_g = 0$.*

Proof. [GJ70a, Theorems 2.2.1 and 2.3.1] □

As partially stated before, simply taking the limit $g \to 1$ in the Fock space $\mathcal{F}_+\big(\mathrm{L}^2(\mathbb{R}^3)\big)$ yields the following problems.

- $E_g \to \infty$ so the Hamlitonian does not converge
- $\Omega_g \to 0$ in the weak sense

The way out of this conundrum is to follow the idea outlined in Remark 2.43 and to define the field dynamics algebraically, rather than in terms of unitaries. The first step is to define a C*-algebra $\mathcal{A}$ in terms of the fields $\Phi(f)$ from Eq. (3.3).

Definition 3.5. *With $\mathcal{H} = \mathcal{F}_+\big(\mathrm{L}^2(\mathbb{R}^2)\big)$ we define $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ as the smallest C*-algebra[17] containing all unitaries $\exp(i\Phi(f))$ for arbitrary $f \in \mathscr{D}(\mathbb{R}^2, \mathbb{R})$; i.e. smooth real-valued functions with compact support.*

Proposition 3.6. *There is a unique one parameter group α_t of automorphisms $\alpha_t : \mathcal{A} \to \mathcal{A}$ satisfying*

$$\alpha_t\big(\exp\big(i\Phi(f)\big)\big) = \exp(itH_g)\exp\big(i\Phi(f)\big)\exp(-itH_g),$$

where $g(x) = 1$ holds on a sufficiently large interval.

Proof. [GJ70a, Sec. 4]. □

This automorphism group describes the time evolution of the interacting fields algebraically. In a similar way we can proceed with the vacuum.

Definition 3.7. *For any $A \in \mathcal{A}$, we define the expectation values*

$$\omega_g(A) = \langle \Omega_g, A\Omega_g \rangle.$$

[17] We are glancing over some technicalities here, because the original construction in [GJ70b] is a bit more involved. However, the simplification we are applying here shouldn't cause big differences.

Theorem 3.8. *The following statements hold.*

(a) ω_g *is a state of* $\mathcal{A}$.

(b) There is a sequence $(g_n)_{n\in\mathbb{N}}$ *of cut-offs such that* $g \to \infty$ *and there is a state* ω *such that* $\lim_{n\to\infty} \omega_{g_n}(A) = \omega(A)$ *for all* $A \in \mathcal{A}$.

Proof. [GJ70b, Theorem 2.1]. □

Having this ω the idea is to get back a field which now should be a Wightman field. The idea is to use the GNS representation of the state ω just constructed. Hence we have a Hilbert space $\mathcal{H}_\omega$, a *-morphism $\pi_\omega : \mathcal{A} \to \mathcal{B}(\mathcal{H}_\omega)$, and a cyclic vector $\Omega_\omega \in \mathcal{H}_\omega$ such that $\omega(A) = \langle \Omega_\omega, \pi_\Omega(A)\Omega_\omega\rangle$.

Theorem 3.9. *There is a positive self-adjoint operator* H_ω *on* $\mathcal{H}_\omega$ *such that*

$$\exp(itH_\omega)\pi_\omega(A)\exp(-itH_\omega) = \pi\big(\alpha_t(A)\big) \quad \textit{and} \quad H_\omega\Omega_\omega = 0 \tag{3.5}$$

Proof. [GJ70b, Theorem 2.1]. □

Remark 3.10. (a) The first statement in (3.5) says that the automorphism group α_t is unitarily implemented in the representation π_ω, and the corresponding unitary group is generated by the operator H_ω which is the *renormalized Hamiltonian* of the interacting theory. It is the limit of H_g in the sense that

$$\langle \pi_w(B)\Omega_w, e^{iH_w t}\pi_w(A)\Omega\rangle$$

is obtained through a limit as $g \to 1$ of

$$\langle B\Omega_g, e^{iH_g t}A\Omega_g\rangle.$$

(b) The GNS vacuum Ω_ω is, again by Eq. (3.5) an eigenvalue of H with eigenvalue 0. Since H is positive, Ω_ω is eigenvector with the lowest energy.

(c) We call Ω_ω the physical vacuum and π_ω the physical representation. The vectors in H_ω are called physical vectors.

Now let us come to the final step, which is the reconstruction of the fields in the new representation. This is done in terms of the following lemma.

Lemma 3.11. *Consider* $f \in \mathscr{S}(\mathbb{R}^2, \mathbb{R})$. *The one parameter unitary group*

$$\mathbb{R} \ni \lambda \mapsto V_\lambda = \pi\omega\big(\exp\big(i\lambda\Phi(f)\big)\big) \in \mathcal{U}(\mathcal{H}_\omega)$$

is strongly continuous.

Proof. This is a consequence of the "locally Fock" property proved in [GJ70b, Thm 2.2]; cf. also the corresponding remarks in Sec. 1 of [GJ70b]. □

Now we can introduce the *renomalized fields* as

$$\Phi_{\mathrm{ren}}(f)\psi = \frac{1}{i}\frac{d}{d\lambda}V_\lambda\psi\Big|_{\lambda=0},$$

with the unitaries V_λ from the preceding lemma. The set of $\psi \in \mathcal{H}_\omega$ for which the limit exists forms the domain of $\Phi_{\mathrm{ren}}(f)$. Now all properties of the field $\Phi(f)$ constructed in Sec. 3.2 can be carried over. To see this we again have to use the locally Fock property of π_ω from [GJ70b, Thm 2.2]. This basically finishes the construction. Let us summarize the result.

Theorem 3.12. *The fields $\Phi_{\mathrm{ren}}(f)$ admit an invariant dense domain $D \subset \mathcal{H}_\omega$, and a strongly continuous repesentation $\mathcal{P}_+^\uparrow \ni (b,\Lambda) \mapsto U(b,\Lambda) \in \mathcal{U}(\mathcal{H}_\omega)$ such that the time translations coincide with $\exp(itH_\omega)$. The five-tuple $(\mathcal{H}_w, D, \Phi_\omega, \Omega_\omega, U)$ is a Wightman quantum field.*

Proof. This is not explicitly shown in [GJ70b] but implicitly discussed [GJ70b, Sec. 1]. Basically, all properties can either be carried over from the fields $\Phi(f)$ by using [GJ70b, Thm 2.2], or they are consequences of related properties of the GNS representation. The only missing point is the representation U. The space translations are treated in [GJ70b] along the same lines as the time evolution $\exp(itH_\omega)$. The whole Poincaré group is treated in [CJ70]. Together with the positivity of H_ω (which generates the time translations) this also shows that the spectrum condition holds. □

Remark 3.13. We again can use Haag's theorem to analyze this construction. Since the Wightman field we got is not the free field, by Theorem 3.3 said field has to be unitarily inequivalent to the free field for all times. Hence, the renormalization process just outlined, resulted in a change of representations of the CCR. The original one was the vacuum representation of the free field, which does not contain a Hamiltonian for the interacting model. The new representation is the vacuum representation of the interacting field and it does not contain the free Hamiltonian. A similar statement holds for the vacuum. The renormalized vacuum Ω_ω can not be described in terms of density operators in the vacuum representation of the free field. Roughly speaking, this is caused by the fact that we can say that this new vacuum contains infinitely many particles of the free theory, and therefore it is not defined in our original Fock space (which only contains states with finitely many particles).

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