Mathematical Aspects of Quantum Field Theory

Michael Keyl (FU Berlin)[1]

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[1]Mail adress: michaelkeyl137@gmail.com
[2]Mail adress: frederik.vom-ende@tum.de

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0 Introduction

In quantum mechanics, the standard approach is as follows. We have a Hilbert space $\mathcal{H}$ (e.g. $\mathcal{H} = \mathrm{L}^2(\mathbb{R}^3)$ for a particle in three dimensions), a Hamiltonian H describing the dynamics (like a Schrödinder operator $H = -\Delta + V(x)$ for a particle moving in a potential V), and the standard observables position Q_j and momentum P_k described by self-adjoint operators on $\mathcal{H}$. In Quantum Field Theory very loosely speaking, the main difference concerns these observables. While we still work on Hilbert spaces and with Hamiltonians, the Q_j and P_k are replaced by field amplitudes $\mathbb{R}^3 \ni x \mapsto \phi(x)$ and field momenta $\mathbb{R}^3 \ni x \mapsto \pi(x)$ at a point (or space time event) x. Mathematically they are usually not described by operators but quadratic forms in $\mathcal{H}$.

There are many reasons to use fields rather than particles. Often both discriptions are equivalent and the corresponding fields admit an interpretation in terms of particles. In such a situation it is a matter of convenience which pictures should be preferred. But in some cases the particle point of view (or more generally spoken: the point of view of ordinary Quantum Mechanics) is too limited and fields are really needed for a consistent description. The most prominent example where this happens is *Relativistic* Quantum Mechanics which does not exist *as a consistent theory*. Problematic are in particular the following issues:

- *Locality.* Consider particle described by a wave function $\Psi \in \mathrm{L}^2(\mathbb{R}^3)$ and located in the region Σ, i.e. $\operatorname{supp}\psi \subset \Sigma$. If we follow the usual rule and describe the probability to detect a particle in $\Sigma \subset \mathbb{R}^3$ by $\int_\sigma |\Psi(x)|^2\,dx$ the particle is located in Σ with certainty. After evolving the wave function freely, and for an arbitrarily short time ϵ (e.g. with the relativistic Hamilton operator $H = \sqrt{P^2 + m^2}$) the new wave function ψ_ϵ has non compact support, i.e. the probability to find the particle arbitrarily far away from its original position is non-zero; cf. Fig. 1. Something like this can also happen in non-relativistic models. In those cases, however, infinite speed is not a conceptual problem. In Special Relativity, on the other hand, it is.

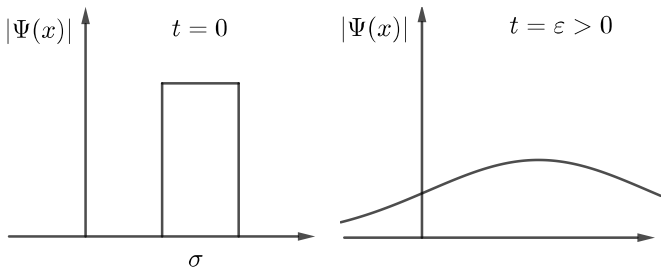


Figure 1: Possible evolution of the amplitude of the wave function for some arbitrary timestep $\varepsilon > 0$. This however is non-consistent with special relativity, in particular the speed of light.

- *Particle creation.* Even in simple setups involving relativistic particles in external potential particle creation effects can occur. If you want to describe this within

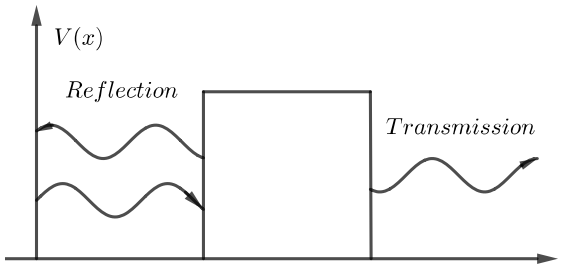


Figure 2: Klein's result showed that if the potential is of the order of the electron mass, the barrier is nearly transparent. Also for the dirac equation, more particles can be reflected than were sent in because of pair production. The number of particles not being fix cannot be described by quantum mechanics.

a model which is based on the assumption that the particle number is fixed this leads to contradictions. A typical example is Klein's paradox where an electron beam hitting a potential barrier is described. Within relativistic Quantum Mechanics we get strange behaviours of reflection and transmission coefficients which can be explained if we involve pair creation; cf. Fig. 2. Note that dynamical changes of the particle number can also occur in non-relativistic models (e.g. in solid state physics). However, in the relativistic case particle creation and annihilation is the rule even in simple cases involving only external potentials and no real interaction. It can be avoided only if we restrict our attention to free particles.

Both problems can be resolved by fields: Firstly, in Quantum Field Theory we can localize fields (observables) rather than wave functions (states), and secondly, fields are perfectly capable to describe particle models where the particle number is not fixed.

0.1 About this Document

These lecture notes are based on a course I (MK) have held in autumn and winter 2017/2018 at LMU and TU in Munich. Most of the writing and editing was done by Frederik vom Ende. In its present form (version 1.0) the document is mostly based on notes taken directly from the black board or my handwritten preparations. Therefore this version has a number of defects (my fault not Frederik's): 1. Some important stuff which I haven't had the time to treat during the lectures (at least not in sufficient detail) is missing. 2. The organization is sometimes suboptimal. 3. The presentation follows sometimes (too) closely the literature I have used during preparation (you might call this plagarism, but this is not a thesis after all ...) The plan is to deal with these problems in future versions. Hence, wherever you have got this file from, you should look there regularly for updated versions.

0.2 Version History

1. **Version 1.0:** This version. Produced during and immediately after the lectures in October 2017 and February 2018.

1 Wightman Quantum Field Theory

1.1 Tempered Distributions

For this section we orient ourselves towards [RS80, Chapter V]. First we introduce the following standard notation. For $f : \mathbb{R}^n \to \mathbb{C}$, $\alpha, \beta \in \mathbb{N}_0^n$ and $\mathbb{R}^n \ni x = (x_1, \ldots, x_n)$ we denote

$$x^\beta = x_1^{\beta_1} \ldots x_n^{\beta_n} \qquad |\alpha| = \sum_{j=1}^{n} \alpha_j \qquad D^\alpha f = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}}.$$

Definition 1.1. *The functions of rapid decrease, also called Schwartz functions, is the set defined as*

$$\mathscr{S}(\mathbb{R}^n) := \{f \in C^\infty(\mathbb{R}^n, \mathbb{C}) \,|\, \|f\|_{\alpha\beta} < \infty \quad \forall_{\alpha,\beta \in \mathbb{N}_0^n}\}$$

where $\|f\|_{\alpha\beta} = \sup_{x \in \mathbb{R}^n} |x^\alpha D^\beta f(x)|$.

Remark 1.2 (Locally convex spaces)**.** Let V be a complex vector space V and $(p_j)_{j \in I}$ a family of seminorms[3] which separates points, so $p_j(f) = 0$ for all $j \in I$ implies $f = 0$. Then $(V, (p_j)_{j \in I})$ is said to be a locally convex space. A neighborhood base at $0 \in V$ is given by the sets

$$\mathcal{N}(j_1, \ldots, j_m; \varepsilon) = \{f \in V \,|\, p_{j_k}(f) < \varepsilon \quad \forall_{k \in \{1,\ldots,m\}}\},$$

which can be regarded as an analogon of the family of ε-balls in Banach spaces. By translating the $\mathcal{N}(j_1, \ldots, j_m; \varepsilon)$ within V we can define similar bases at each $v \in V$. In this way V becomes a topological space, and addition and scalar multilpication become continuous maps. In other words, V equipped with topology is a topological vector space. If $J = \mathbb{N}$, then we can define a metric d via

$$d(f, g) = \sum_{n=1}^{\infty} 2^{-n} \frac{p_n(f - g)}{1 + p_n(f - g)}$$

which generates the same topology as before. If $(V, (p_n)_{n \in \mathbb{N}})$ is complete, then it is called Fréchet space.

The following is an easy application of the neighbourhood definition given above.

Lemma 1.3. *A linear functional $\phi : V \to \mathbb{C}$ is continuous if and only if one can find a finite set $j_1, \ldots, j_m \in I$ of seminorms and $C \in \mathbb{R}_0^+$ such that*

$$|\phi(x)| \leq C(p_{j_1}(x) + \ldots + p_{j_m}(x))$$

for each $x \in V$.

[3] A seminorm is a norm which is not definite so for $f, g \in V$ and $\lambda \in \mathbb{C}$ we have $p(\lambda f) = |\lambda| p(f)$ and $p(f + g) \leq p(f) + p(g)$, but $p(f) = 0$ does not necessarily imply $f = 0$.

Proposition 1.4 ([RS80], Theorem V.9). *The vector space $\mathscr{S}(\mathbb{R}^n)$ with the natural topology given by the seminorms $\|\cdot\|_{\alpha,\beta}$ is a Fréchet space.*

Definition 1.5. *The topological dual[4] $\mathscr{S}'(\mathbb{R}^n)$ of $\mathscr{S}(\mathbb{R}^n)$ is called the space of tempered distributions.*

Example 1.6. 1. (δ-distribution). Consider $\delta_x : \mathscr{S}(\mathbb{R}^n) \to \mathbb{C}$ for some $x \in \mathbb{R}^n$ where $f \mapsto \delta_x(f) := f(x)$. We then have

$$|\delta_x(f)| \leq \sup_{y\in\mathbb{R}^n} |f(y)| = \|f\|_{0,0},$$

so $\delta_x \in \mathscr{S}'(\mathbb{R}^n)$. Sometimes it is useful to use the formal expression

$$f(x) = \int_{\mathbb{R}^n} \delta(x-y) f(y)\, dy$$

which involves the "delta-function" δ. When we do this it is important to keep in mind that δ really does not exist as a function, i.e. we can not evaluate it at each $x \in \mathbb{R}$, we can only evaluate it "under the integral".

2. (Measures). Consider a finite Borel measure μ, so we can define $\mu : \mathscr{S}(\mathbb{R}^n) \to \mathbb{C}$, $f \mapsto \int_{\mathbb{R}^n} f(x)\mu(dx)$. Continuity of this map can be shown analogously to the δ-distribution case, so $\mu \in \mathscr{S}'(\mathbb{R}^n)$.
3. Let us look at this special case of the second example. For $g \in \mathscr{S}(\mathbb{R}^n)$ we can define $\phi_g(f) = \int_{\mathbb{R}^n} f(x)g(x)\, dx$. Moreover, if $g_1 \neq g_2$ as functions in $\mathscr{S}$, then $\phi_{g_1} \neq \phi_{g_2}$. This embeds $\mathscr{S}$ naturally in $\mathscr{S}'$.
4. Similarly, for $g \in L^p(\mathbb{R}^n)$ and $p \in \mathbb{N}$ we have

$$\phi_g(f) = \int_{\mathbb{R}^n} f(x)g(x)\, dx$$

which embeds $L^p \hookrightarrow \mathscr{S}'$ in a similar way.

Remark 1.7. We equip $\mathscr{S}'(\mathbb{R}^n)$ with the weak-$*$-topology, which is generated by seminorms $f \mapsto |\phi(f)|$ with $\phi \in \mathscr{S}'(\mathbb{R})$. Then, the linear subset $\mathscr{S}(\mathbb{R}^n) \subset \mathscr{S}'(\mathbb{R}^n)$ is dense and the embedding $\iota : \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}'(\mathbb{R}^n)$ is continuous. This suggests extending continuous maps $T : \mathscr{S} \to \mathscr{S}$ to $\mathscr{S}'$ as follows. If $T : \mathscr{S} \to \mathscr{S}$ is continuous, then $\iota \circ T : \mathscr{S} \to \mathscr{S}'$ is continuous as well by continuity of ι. Since $\mathscr{S}$ is dense is $\mathscr{S}'$, there is at most one continuous extension of $\iota \circ T$. To find this extension we look for a continuous $S : \mathscr{S} \to \mathscr{S}$ with the adjoint $S' : \mathscr{S}' \to \mathscr{S}'$, $\phi \mapsto S'(\phi)$ and ask S' to satisfy $S'(\phi) = \phi \circ S$. Now S' is well-defined and continuous in the weak-$*$-topology. Hence S' is the extension we are looking for and it can be expressed as $T\phi(f) = \phi(S(f))$. In other words the general strategy is to apply the adjoint of T to the test function (assuming that the double adjoint becomes T again).

[4] The topological dual is the space of continuous linear functionals acting on the vector space.

Example 1.8. 1. Consider a C^∞-function $F : \mathbb{R}^n \to \mathbb{C}$ which derivatives are polynomially bounded, so there exists $C \in \mathbb{R}_0^+$ and $n \in \mathbb{N}$, such that

$$|D^\alpha F(x)| \leq C(1 + \|x\|^2)^n$$

for all $\alpha \in \mathbb{N}_0^n$. This means for any $f \in \mathscr{S}(\mathbb{R}^n)$ we have $Ff \in \mathscr{S}(\mathbb{R}^n)$ where $f \mapsto Ff$ is continuous. The extension in this case then is given by $(F\phi)f = \phi(Ff)$. Then

$$\phi_g(Ff) = \int_{\mathbb{R}^n} g(x)(F(x)f(x))\,dx = \int_{\mathbb{R}^n} (g(x)F(x))f(x)\,dx = \phi_{Fg}(f)$$

2. Weak derivative. To extend D^α to $\mathscr{S}'$, partial integration implies that $(D^\alpha\phi)(f) = (-1)^{|\alpha|}\phi(D^\alpha f)$.
3. For the Fourier transform, we simply have $\hat{\phi}(f) = \phi(\hat{f})$.
4. $f(\cdot) \to f(\cdot - a)$ translation, $f(\cdot) \to f(A\cdot)$ with $A \in \mathrm{GL}(n, \mathbb{R})$

Example 1.9 (Heaviside function)**.** Defining

$$\nu(x) = \begin{cases} x & x \geq 0 \\ 0 & x \leq 0 \end{cases},$$

we get the Heaviside function via

$$\Big(\frac{d}{dx}\phi_\nu\Big)(f) = -\phi_\nu(f') = -\int_0^\infty xf'(x)\,dx = \int_0^\infty f(x)\,dx,$$

which implies $\frac{d}{dx}\phi_\nu = \phi_\theta$, where

$$\theta(x) = \begin{cases} x & 1 \geq 0 \\ 0 & x \leq 0 \end{cases}.$$

Further

$$\Big(\frac{d}{dx}\phi_\theta\Big)(f) = -\phi_\theta(f') = -\int_0^\infty f'(x)\,dx = f(0)$$

so $\frac{d}{dx}\phi_\theta = \delta_0$.

Theorem 1.10 (Regularity theorem for distributions, [RS80] Theorem V.10)**.** *Let $\phi \in \mathscr{S}'(\mathbb{R}^n)$. Then $\phi = D^\beta g$ for some polynomially bounded continuous function $g : \mathbb{R}^n \to \mathbb{C}$ and some $\beta \in I_+^n$, that is,*

$$\phi(f) = \int (-1)^{|\beta|} g(x)(D^\beta f)(x) d^d x$$

for all $f \in \mathscr{S}(\mathbb{R}^n)$.

Definition 1.11 (Support)**.** *We say $\phi \in \mathscr{S}'(\mathbb{R}^n)$ vanishes on an open subset $\sigma \subset \mathbb{R}^n$ if $\phi(f) = 0$ for all f with $\operatorname{supp}(f) \subset \sigma$. Then $\operatorname{supp}\phi$ is the complement of the largest open set on which ϕ vanishes.*

Definition 1.12. *Let $\phi \in \mathscr{S}'(\mathbb{R}^n)$. We say $x \in \mathbb{R}^n$ is a regular point of ϕ if there exists an open neighbourhood U of x and $g \in C^\infty(U)$, such that $\phi(f) = \phi_g(f)$ for all f with $\operatorname{supp} f \subset U$. The complement of the set of regular points is called the singular support.*

An easy example is the singular support of the δ-distribution, which obviously only is the zero.

Theorem 1.13 (Kernel or nuclear theorem, [RS80] Theorem V.12)**.** *Let $B(f,g)$ be a separately continuous bilinear functional on $\mathscr{S}(\mathbb{R}^n) \times \mathscr{S}(\mathbb{R}^m)$. Then there is a unique tempered distribution $T \in \mathscr{S}'(\mathbb{R}^{n+m})$ with $B(f,g) = T(f \otimes g)$ where*

$$(f \otimes g)(x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+m}) = f(x_1, \ldots, x_n) g(x_{n+1}, \ldots, x_{n+m}).$$

The result can easily be extended to more than two tensor factors.

Remark 1.14 (Distributions vs. tempered distributions)**.** To get a general distribution we have to replace Schwartz functions as test functions by smooth compactly supported functions. One big advantage of this approach is that we can define (general) distributions on arbitrary open subsets $\Omega \subset \mathbb{R}^n$, not only on $\mathbb{R}^n$ itself. We define

$$\mathscr{D}(\Omega) = \{f : \Omega \to \mathbb{C} \,|\, f \in C^\infty \text{ and } \operatorname{supp} f \text{ is compact}\}.$$

We need this in Chapter 5 where we have to restrict distributions to open sets. The space $\mathscr{D}(\Omega)$ can be equipped with a topology which transforms it into a locally convex space. This is done via an inductive limit construction which we skip here (cf. Sect. V.4 of [RS80] instead). We will directly proceed to the next defintion.

Definition 1.15. *A generalized function or distribution is a continuous linear functional on $\mathscr{D}(\Omega)$. The space of all continuous linear functionals on $\mathscr{D}(\Omega)$ is denoted by $\mathscr{D}'(\Omega)$.*

Since we haven't presented the definition of the topology on $\mathscr{D}(\Omega)$ we give the following alternative characterization of continuity (cf. Sect. V.4 of [RS80]).

Proposition 1.16. *A linear functional T on $\mathscr{D}(\mathbb{R}^n)$ is continuous if and only if for each compact $K \subset \mathbb{R}^n$, there is a constant C and an integer j such that*

$$|T(f)| \leq C \sum_{|\alpha| \leq j} \|D^\alpha f\|_\infty$$

for all $f \in C_0^\infty(K)$.

Remark 1.17 (Distributions vs. tempered distributions II)**.** Tempered distributions in particular are distributions, i.e. $\mathscr{S}'(\mathbb{R}^n) \subset \mathscr{D}'(\mathbb{R}^n)$. They are, however, restricted to be polynomially bounded at infinity, while general distributions can have an unrestricted

growth. The main advantage of this growth restriction is the possibility to define the Fourier transform for elements of $\mathscr{S}'(\mathbb{R}^n)$; cf. Example 1.8. This is not possible for general distributions. The latter, however, allow an easy definition of restrictions to open subsets $\Omega \subset \mathbb{R}^n$. The following statement is readily verified with Proposition 1.16.

Proposition 1.18. *Consider open sets $A \subset \mathbb{R}^n$ and $\Omega \subset A$ as well as $T \in \mathscr{D}'(A)$. The map $T|_\Omega : \mathscr{D}(\Omega) \to \mathbb{C}$ is an element of $\mathscr{D}'(\Omega)$ called the* restriction *of T.*

Remark 1.19 (Distributions with compact support)**.** A distribution $T \in \mathscr{D}'(\Omega)$ with compact support $\operatorname{supp} T \subset \Omega$ can be easily extended to $\mathbb{R}^n$. We only need a smooth function g satisfying $g(x) = 1$ on $\operatorname{supp} T$ and $g(x) = 0$ outside Ω. Then we define $\tilde{T}(f) = T(gf)$ for any test function $f \in \mathscr{D}(\mathbb{R}^n)$. It is easy to see that this is a distribution in $\mathscr{D}'(\mathbb{R}^n)$ and since it is still compactly supported (obviously $\operatorname{supp} T \subset \operatorname{supp} \tilde{T}$) we even get $\tilde{T} \subset \mathscr{S}'(\mathbb{R}^n)$. Hence we can Fourier transform T (dropping the tilde) and it turns out that $\operatorname{supp} T$ is compact iff $\hat{T}$ is an entire analytic function. Hence, for a general distribution $T \in \mathscr{D}'(\Omega)$ we can choose a compactly supported function $f \in \mathscr{D}(\Omega)$ and look at fT. Its Fourier transform $\widehat{fT}$ then is a smooth function on $\mathbb{R}^n$. This method will be used in the following definition.

Definition 1.20. *Let $T, S \in \mathscr{D}'$. We say that $W \in \mathscr{D}'$ is the product of T and S iff for each $x \in \mathbb{R}^n$, there exists some $f \in \mathscr{D}$ with $f = 1$ near x so that for each $k \in \mathbb{R}^n$*

$$\widehat{f^2 W}(k) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \widehat{fT}(l)\widehat{fS}(k-l)\, dl$$

where the integral is absolutely convergent. If such W exists, we say the product of T and S exists.

Products of distributions will become very important for the discussion of perturbation theory in Chapter 5.

1.2 Quantum Fields

Let us, as a reminder, quickly go over some notation and results regarding unbounded operators. For this, we orient ourselves towards [RS80, Chapter VIII]. Here, $\mathcal{H}$ is any separable Hilbert space.

- An operator is a linear map $A : D(A) \to \mathcal{H}$. The subspace $D(A) \subset \mathcal{H}$ is the domain of A. If $D(A)$ is dense, then the operator is called densely defined.
- The graph of A is defined to be $\Gamma(A) = \{(x, Ax) \,|\, x \in D(A)\} \subset \mathcal{H} \times \mathcal{H}$. Then A is called closed if $\Gamma(A) \subset \mathcal{H} \times \mathcal{H}$ is closed.
- An operator B is said to be an extension of A if $\Gamma(A) \subset \Gamma(B)$, we then write $A \subset B$.
- An operator A is closable if it has a closed extension. If A is closable, it has a smallest closed extension $\overline{A}$ characterized by $\Gamma(\overline{A}) = \overline{\Gamma(A)}$. We then say $\overline{A}$ is the closure of A.

- Let A be densely defined. We define

$$D(A^*) = \{x \in \mathcal{H} \mid \exists_{y \in \mathcal{H}} \forall_{z \in D(A)} : \langle Az, x\rangle = \langle z, y\rangle\}$$

  and for $x \in D(A^*)$ we define the adjoint via $A^*x := y$ for respective y. Note that $D(A^*) = \{0\}$ is possible.
- Let A be densely defined. Then A is closable if and only if $D(A^*)$ is densely defined. In that case, A^* is closed and the double adjoint satisfies $A^{**} = \overline{A}$.
- A is said to be symmetric or hermitian if $\langle Ax, y\rangle = \langle x, Ay\rangle$ for all $x, y \in D(A)$. This is equivalent to $D(A) \subset D(A^*)$ with $Ax = A^*x$ for all $x \in D(A)$. Also, A is called self-adjoint if $A = A^*$. In particular, this means $D(A) = D(A^*)$.
- If A is symmetric, then $A \subset A^{**} \subset A^*$. If A is symmetric and closed, then $A = A^{**} \subset A^*$. If A is self-adjoint, then $A = A^{**} = A^*$.
- We call A essentially self-adjoint if A is symmetric and $\overline{A}$ is self-adjoint.

For the topic of quantum fields, we refer to [RS75, Chapter IX.8].

Definition 1.21. *A quantum field is a 4-tuple $(\mathcal{H}, D, \Phi, \Omega)$ consisting of a (separable) Hilbert space $\mathcal{H}$, a dense subspace $D \subset \mathcal{H}$, a $\mathbb{C}$-linear map*

$$\Phi : \mathscr{S}(\mathbb{R}^n) \to \mathcal{L}(D, \mathcal{H}),$$

where $\mathcal{L}(D, \mathcal{H})$ denotes the linear maps between D and $\mathcal{H}$, and an element $\Omega \in D \subset \mathcal{H}$, such that the following conditions hold.

(a) D is invariant, so $\Phi(f)D \subset D$ for all $f \in \mathscr{S}(\mathbb{R}^n)$.

(b) Ω is cyclic which means that the set

$$D_0 := \{\Phi(f_1) \dots \Phi(f_m)\Omega \mid f_1, \dots, f_m \in \mathscr{S}(\mathbb{R}^n), m \in \mathbb{N}\} \subset D \subset \mathcal{H}$$

is dense in $\mathcal{H}$.

(c) $\Phi(f)$ is closable for all $f \in \mathscr{S}(\mathbb{R}^n)$.

(d) For all $x, y \in D$, the map

$$\mathscr{S}(\mathbb{R}^n) \ni f \longmapsto \langle x, \Phi(f)y\rangle$$

is a tempered distribution.

Note that D_0 can replace D since D_0 is also dense in $\mathcal{H}$.

Definition 1.22. *A quantum field $(\mathcal{H}, D, \Phi, \Omega)$ is called hermitian if*[5]

$$\Phi(\overline{f}) = \Phi(f)^*|_D$$

and if for $f \in \mathscr{S}(\mathbb{R}^n, \mathbb{R})$ the operator $\Phi(f)$ is essentially self-adjoint on D.

[5] $\overline{f}$ denotes the complex conjugate of f

Remark 1.23 (Interpretation)**.** A hermitian quatum field describes a special observable, or more precisely a whole family of observables. To understand this remark a bit better let us pretend that we can write $\Phi(f)$ as

$$\Phi(f) = \int_{\mathbb{R}} f(x)\Phi(x)\,dx \tag{1.1}$$

with self-adjoint operators $\Phi(x)$, $x \in \mathbb{R}$. Note that this is usually not possible and we will discuss in Ch. 2 how Eq. (1.1) can be interpreted in a mathematical rigorous way. For now we just look at the operator-valued field $\Phi(x)$ as a formal expression for an operator-valued distribution in the same way as the delta-function $\delta(x)$ is a formal expression for the delta-distribution δ_0. In doing so we can look at $\Phi(x)$ as the observable which measures the *field amplitude* in $x \in \mathbb{R}^n$. The smeared out version $\Phi(f)$ from Eq. (1.1) can then be regarded as the *averaged* field amplitude with averaging function f. This interpretation makes it very clear that the $\Phi(f)$ are *local* observables, i.e. they can be measured in any region containing the support of f.

To keep this interpretation, self-adjointness of $\Phi(f)$ for real-valued f is mandatory. Mathematically, however, this self-adjointness condition is often annoying, since it is difficult to prove. Most others drop it therefore and accept hermitian quantum fields where the $\Phi(f)$ are really only hermitian. We will see in the next section why this can be advantageous.

Remark 1.24. Up to now we can generalize everything to manifolds by replacing $\mathscr{S}(\mathbb{R}^n)$ with

$$D(M) = \{f : M \to \mathbb{C} \,|\, f \text{ is } C^\infty \text{ and compactly supported}\}$$

for some C^∞-manifold M.

Remark 1.25 (Poincaré group)**.** The definitions presented so far are fairly general and lack in particular any dynamical content. This will change now, when we study quantum fields in Minkowski space. For that, let us have a short recap on some concpets and notations.

- The Minkowski metric on $\mathbb{R}^4$ is given by

$$\eta(v, w) = v^0 w^0 - \sum_{j=1}^{3} v^j w^j.$$

- The Lorentz group is defined to be

$$O(3,1) = \{\Lambda \in \mathrm{GL}(4,\mathbb{R}) \,|\, \eta(\Lambda v, \Lambda w) = \eta(v, w)\}.$$

  The restricted Lorentz group then is

$$\mathrm{SO}^\uparrow(1,3) = \{\Lambda \in O(3,1) \,|\, \det\Lambda = 1, \langle e_0, \Lambda e_0\rangle > 0\}.$$

- The restricted Poincaré group $\mathcal{P}_+^\uparrow$ is the semi-direct product $\mathbb{R}^4 \rtimes \mathrm{SO}^\uparrow(1,3)$. Hence $\mathcal{P}_+^\uparrow$ is the set of pairs (a, Λ) where $a \in \mathbb{R}^4$ and $\Lambda \in \mathrm{SO}^\uparrow(1,3)$, and the group operation is

$$(a, \Lambda_1)(b, \Lambda_2) = (a + \Lambda_1 b, \Lambda_1 \Lambda_2).$$

  The Poincaré group describes transformations from one inertial system into another by the coordinate transformation $v \mapsto a + \Lambda v$.

- Now assume that $\mathcal{P}_+^\uparrow \ni (a, \Lambda) \mapsto U(a, \Lambda) \in \mathcal{U}(\mathcal{H})$, with the unitary group $\mathcal{U}(\mathcal{H})$ on $\mathcal{H}$, is a strongly continuous[6], unitary representation of $\mathcal{P}_+^\uparrow$ on $\mathcal{H}$. By strong continuity we can define generators of the translations

$$P_j \xi = -i \, \frac{d}{d\lambda} U(\lambda e_j, \mathbb{1}) \xi \Big|_{\lambda=0}$$

  for $j = 0, \ldots, 3$ with standard basis $(e_j)_j$ of $\mathbb{R}^4$. Here $\xi \in \mathcal{H}$ is chosen such that the limit $\lambda \to 0$ exists. Note that $U(\lambda e_j, \mathbb{1})\xi$ is a time translation for $j = 0$ and a space translation if $j = 1, 2, 3$. Hence P_0 is the Hamiltonian and $P_1, \ldots, P_3$ are momentum operators. With an arbitrary $a \in \mathbb{R}^4$ we get

$$U(a, \mathbb{1}) = \exp\Big(i \sum_{j=0}^{3} a_j P_j\Big). \tag{1.2}$$

- If $a = \Lambda e_0$ with a Lorentz transfomation Λ, the generator P_a of the one parameter group $\mathbb{R} \ni t \mapsto U(ta, \mathbb{1}) \in \mathcal{U}(\mathcal{H})$ is given by

$$P_a = U(0, \Lambda) P_0 U(0, \Lambda)^* = \sum_{j=0}^{3} a_j P_j$$

  In other words, P_a is the Hamiltonian the inertial observer with four-velocity a is seeing. The Lorentz transformation Λ describes the transition from "our" intertial system (i.e. where we are at rest) into the one of observer a. Please note that according to this reasoning the representation $U(a, \Lambda)$ does not only describes the transition between inertial systems, but also contains the *complete description of the dynamical structure of our theory.*

- Since the translations form an abelian subgroup, the unitaries $U(a, \mathbb{1})$ mutually commute and can therefore be jointly "diagonalized". More precisely, by the spectral theorem ([RS80], Ch. VII and Sec. VIII.3), there is a projection valued measure[7]

$$E : \mathfrak{B}(\mathbb{R}^4) \to \mathcal{B}(\mathcal{H}) \qquad \Sigma \longmapsto E(\Sigma)$$

[6] This means that $\mathcal{P}_+^\uparrow \ni (\alpha, \Lambda) \mapsto U(\alpha, \Lambda)\xi \in \mathcal{H}$ is continuous for all $\xi \in \mathcal{H}$.

[7] Here, $\mathfrak{B}(\mathbb{R}^4)$ denotes the σ-algebra of Borel subsets of $\mathbb{R}^4$ and $\mathcal{B}(\mathcal{H})$ are the bounded linear operators on $\mathcal{H}$.

on $\mathbb{R}^4$ such that

$$\langle \xi, U(a, \mathbb{1})\xi \rangle = \int_{\mathbb{R}^4} \exp\left(i \sum_j a_j \lambda_j\right) \langle \xi, E(d\lambda)\xi \rangle. \tag{1.3}$$

- Readers which are unfamiliar with measure theory should think of $\langle \xi, E(d\lambda)\xi \rangle$ as $E_\xi(\lambda)\, d\lambda$ with singular (i.e. "distributional") density function $E_\xi(\lambda)$. The latter can be written as $E_\xi(\lambda) = \langle \xi, E(\lambda)\xi \rangle$ with a likewise singular map E from $\mathbb{R}^4$ into the set of projection operators in $\mathcal{H}$. Note, however, that this is a very handwaving point of view. A mathematical rigorous interpretation *requires* measure theory.
- Finally we define two regions σ_1, σ_2 to be *spacelike separated* if we can not reach σ_1 from σ_2 with a causal curve[8] and vice versa. This is best described in a picture; cf. Figure 3.

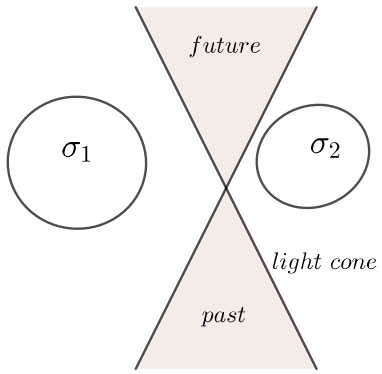


Figure 3: Two open regions $\sigma_1, \sigma_2 \subset \mathbb{R}^4$ are called spacelike separated if they are separated by the light cone so there is no physically possible information exchange between them. Formally this means that for any $x \in \sigma_1$, $y \in \sigma_2$ we have $\eta(x-y, x-y) < 0$.

Definition 1.26. *A 5-tuple $(\mathcal{H}, D, \Phi, \Omega, U)$ consisting of a hermitian quantum field $(\mathcal{H}, D, \Phi, \Omega)$ and a strongly continuous representation of the Poincaré group U is called Wightman quantum field if the following conditions are fulfilled.*

(a) (Local commutativity or microscopic causality). If f and g in $\mathscr{S}(\mathbb{R}^4)$ have supports which are spacelike separated, then $[\Phi(f), \Phi(g)] = 0$.

(b) (Special covariance). For each $(a, \Lambda) \in \mathcal{P}_+^\uparrow$ and all $f \in \mathscr{S}(\mathbb{R}^4)$ we have

$$\Phi((a, \Lambda), f) = U(a, \Lambda)\Phi(f)U(a, \Lambda)^*,$$

where $((a, \Lambda), f)(x) = f(\Lambda^{-1}(x - a))$.

(c) (Uniqueness and invariance of the vacuum). There exists a unique vector $\Omega \in \mathcal{H}$ such that for all $a \in \mathbb{R}^4$ we have

$$U(a, \mathbb{1})\Omega = \Omega.$$

[8] This means by travelling with at most the speed of light.

(d) (Spectral condition). The support of the spectral measure E from Eq. (1.3) is contained in the forward light cone $V^+ = \{v \in \mathbb{R}^4 \,|\, V_0 > 0,\ \eta(v,v) > 0\}$.

Remark 1.27. The local commutativity condition mathematically expresses the quantum-mechanical statement that measurements in spacelike separated regions should be jointly measurable; cf. the discussion in Remark 1.23. Note that commutation in the given form does not guarantee jointly measurability, but it is a necessary condition; cf. in this context [RS80], Sec. VIII.5.

Remark 1.28 (Spectrum condition)**.** The spectral condition means that the joint spectrum of P_j is contained in the forward light cone, so $\sigma(P_0) \subset \mathbb{R}_0^+$ and the energy is positive for all inertial systems. This can be seen very easily and without advanced knowledge if we assume that all the P_j have purely discrete spectrum. Note that this is not very realistic from the physical point of view since the four momenta usually have continuous spectrum and the three-momentum typically has no eigenvalues at all. It is, however, a very simple case where measure theory is not required and therefore it can help to understand the role of the spectral measure E and the spectrum condition.

Since the P_j are mutually commuting, there exists a complete orthonormal system $(\phi_n)_{n\in\mathbb{N}}$ in $\mathcal{H}$ with $P_j\phi_n = \lambda_{jn}\phi_n$ for $j = 0,\dots,3$. Defining $\lambda_n = (\lambda_{0n},\dots,\lambda_{3n})$ we get

$$\exp\Big(i\sum_{j=0}^{3} a_j P_j\Big)\phi_n = \exp(ia\cdot\lambda_n)\phi_n \implies \exp\Big(i\sum_{j=0}^{3} a_j P_j\Big) = \sum_{n=0}^{\infty} e^{ia\cdot\lambda_n}|\phi_n\rangle\langle\phi_n|.$$

where the sum is strongly convergent[9]. With this and (1.2), we see that the matrix elements of the unitaries are given by

$$\langle x, U(a,\mathbb{1})x\rangle = \sum_{n=0}^{\infty} e^{ia\cdot\lambda_n}|\langle x,\phi_n\rangle|^2.$$

Written as an integral, we get

$$\langle x, U(a,\mathbb{1})x\rangle = \int_{\mathbb{R}^4} e^{ia\cdot\lambda} \underbrace{\sum_{n=0}^{\infty}\delta(\lambda-\lambda_n)|\langle x,\lambda_n\rangle|^2}_{E_x(\lambda)}\, d\lambda.$$

Another option is to use $E_\lambda = \sum_{n=0}^{\infty}\delta(\lambda-\lambda_n)|\phi_n\rangle\langle\phi_n|$ which yields

$$\langle x, U(a,\mathbb{1})x\rangle = \int_{\mathbb{R}^4} e^{ia\cdot\lambda} d\langle x, E_\lambda x\rangle.$$

This expression also makes sense in the non-discrete case. The support of the spectral

[9] This means that $\sum_{n=0}^{\infty} e^{ia\cdot\lambda_n}|\phi_n\rangle\langle\phi_n,\xi\rangle$ converges for all $\xi\in\mathcal{H}$

measure E obviously is given by

$$\sigma(E) = \{\lambda_n \,|\, n \in \mathbb{N}\} \subset \mathbb{R}^4.$$

The spectral condition now demands

$$\sigma(E) \subset V_+ = \{v \in \mathbb{R}^4 \,|\, \eta(v,v) > 0, v^0 > 0\},$$

where V_+ is the forward light cone. Now $\lambda_n \in V_+$ by definition means $\lambda_{0n} > 0$, which implies positive energy.

Remark 1.29 (Interpretation)**.** A Wightman quantum field describes a physical system which transforms covariantly under Poincaré tranformations (i.e. change of inertial systems). It combines the unitary representation $U(a, \Lambda)$ of the Poincaré group, and all objects and concepts derived from it, like the four-momentum operators P_j, $j = 0, \ldots, 3$ (cf. Remark 1.25), with a Hermitian quantum field $\Phi(f)$. The P_j contain in particular the dynamical description of the model. As observables they are not that important from a practical point of view, since they are *global*, i.e. they measure the four-momentum of the whole universe. The field operators $\Phi(f)$ on the other hand are of *local* nature, as pointed out in Remark 1.23. Hence they are more realisitically linked to quantities which actually *can* be measured in an experiment. A typical model involving quantum fields usually contains more than one field, describing different physical quantities or observables. For example in addition to the field Φ describing field amplitudes we might want to look at components of the energy momentum tensor, which are described by additional fields within the same model.

The previous remark might create the impression that the fields are completely kinematical objects, while the dynamics is exclusively contained in the representation $U(a, \Lambda)$. This point of view, however, is wrong. Since a Wightman field is a spacetime field, it does contain dynamical information. The axioms in Definition 1.26, in particular the Poincaré covariance and the invariance of the vacuum, link the field Φ and the representation $U(a, \Lambda)$ very closely together, in other words they are not independent, and the fields contain informations about the representation $U(a, \Lambda)$.

Remark 1.30 (Wightman axioms)**.** We haven't explicitly talked about the Wightman axioms for a scalar field, because we have distributed them over Definitions 1.21, 1.22 and 1.26. In Def. 1.21 and 1.22 we find *(numerations taken from [RS75], Sec. IX.8)*: Invariant domain for the fields (Axiom 4), regularity of the fields (Axiom 5) and the cyclicity of the vaccuum (Axiom 8). In Definition 1.26 we have the existence of the representation U ("relativistic invariance of states"; Axiom 1), the spectral condition (Axiom 2), the invariance of the vacuum ("existence and uniqueness of the vaccum"; Axiom 3), the Poincaré invariance of the fields (Axiom 6) and the local commutativity (Axiom 7).

1.3 Wightman Distributions

Consider a hermitian quantum field $(\mathcal{H}, D, \Phi, \Omega)$ and define

$$\tilde{W}^{(m)} : \underbrace{\mathscr{S}(\mathbb{R}^n) \times \ldots \times \mathscr{S}(\mathbb{R}^n)}_{m \text{ factors}} \to \mathbb{C} \qquad (f_1, \ldots, f_m) \longmapsto \langle \Omega, \Phi(f_1) \ldots \Phi(f_m) \Omega \rangle$$

for any $m \in \mathbb{N}$. By regularity of the field, $\tilde{W}^{(m)}$ is separately continuous in all arguments because

$$f_1 \longmapsto \langle \underbrace{\Omega}_{\in D}, \Phi(f_1) \underbrace{\ldots \Phi(f_m)\Omega}_{\in D} \rangle$$

is continuous, the same is true for

$$f_2 \longmapsto \langle \underbrace{\Phi(f_1)^*\Omega}_{\in D}, \Phi(f_2) \underbrace{\ldots \Phi(f_m)\Omega}_{\in D} \rangle$$

and so forth. By the nuclear theorem *(see Theorem 1.13)*, there exists a unique distribution $W^{(m)} \in \mathscr{S}'(\mathbb{R}^n \times \ldots \times \mathbb{R}^n) = \mathscr{S}'(\mathbb{R}^{n \cdot m})$ such that

$$W^{(m)}(f_1 \otimes \ldots \otimes f_m) = \tilde{W}^{(m)}(f_1, \ldots, f_m).$$

In particular, $\tilde{W}^{(m)}$ is jointly continuous.

Definition 1.31. *The $W^{(m)}$, $m \in \mathbb{N}$ are called Wightman distributions of the quantum field $(\mathcal{H}, D, \Phi, \Omega)$.*

The tasks now are to reconstruct a quantum field from $W^{(m)}$ and to translate the Wightman axioms into conditions on $W^{(m)}$. The solution to both tasks is known as the Wightman reconstruction theorem

Definition 1.32. *A complex vector space $\mathcal{A}$ is called unital $*$-algebra if $\mathcal{A}$ is equipped with a bilinear, associative product,*

$$\mathcal{A} \times \mathcal{A} \ni (A, B) \longmapsto AB \in \mathcal{A}$$

and an antilinear[10] involution[11] ($*$-operation)

$$\mathcal{A} \ni A \longmapsto A^* \in \mathcal{A}$$

which satisfies $(AB)^* = B^*A^*$ and there exists a unit $\mathbb{1} \in \mathcal{A}$ with $A\mathbb{1} = \mathbb{1}A = A$ for all $A \in \mathcal{A}$.

Definition 1.33. *A functional $\omega : \mathcal{A} \to \mathbb{C}$ is called a state of $\mathcal{A}$ if*

[10] This means $(A + \lambda B)^* = A^* + \overline{\lambda} B^*$.
[11] This means $A^{**} = A$.

(a) ω *is linear and continuous.*

(b) (Positivity). $\omega(A^*A) \geq 0$ *for all* $A \in \mathcal{A}$.

(c) (Normalization). $\omega(\mathbb{1}) = 1$.

Lemma 1.34. *A state* ω *of a *-algebra* $\mathcal{A}$ *has the following properties*

(a) (Symmetry). $\omega(A^*B) = \overline{\omega(B^*A)}$.

(b) (Cauchy-Schwarz). $|\omega(A^*B)|^2 \leq \omega(A^*A)\omega(B^*B)$.

Proof idea. Look at $\omega((\lambda A+B)^*(\lambda A+B)) \geq 0$. Otherwise cf. [BR02] Lemma 2.3.10. □

Definition 1.35. *A cyclic representation of a* $*$*-Algebra is a 4-tuple* $(\mathcal{H}, D, \pi, \Omega)$ *consisting of a Hilbert space* $\mathcal{H}$*, a dense subspace* $D \subset \mathcal{H}$*, a vector* $\Omega \in D$ *and a complex linear map*

$$\pi : \mathcal{A} \to \mathcal{L}(D, D) \subset \mathcal{L}(D, \mathcal{H}),$$

such that the following holds for all $A, B \in \mathcal{A}$.

(a) $\pi(AB) = \pi(A)\pi(B)$.

(b) $\pi(A^*) = \pi(A)^*|_D$.

(c) (Cyclicity). $\{\pi(A)\Omega \,|\, A \in \mathcal{A}\} = D_0 \subset D \subset \mathcal{H}$ *is dense.*

Again, D_0 is automatically an allowed domain and actually the smallest one.

Theorem 1.36 (Gelfand–Naimark–Segal (GNS)-representation)**.** *Let* $\mathcal{A}$ *be* $*$*-algebra and* $\omega : \mathcal{A} \to \mathbb{C}$ *be a state. Then there exists a cyclic representation* $(\mathcal{H}_\omega, D_\omega, \pi_\omega, \Omega_\omega)$*, such that*

$$\omega(A) = \langle \Omega_\omega, \pi_\omega(A)\Omega_\omega \rangle \tag{1.4}$$

holds for all $A \in \mathcal{A}$*. If* $D_\omega = D_0$*, this representation is unique up to unitary equivalence.*

Proof. On $\mathcal{A}$ we define

$$\langle A, B \rangle = \omega(A^*B)$$

which is sesquilinear and positive semi-definite by Lemma 1.34, but $\omega(A^*A) = 0$ may happen for $A \neq 0$. Therefore we define

$$\mathscr{I}^\omega = \{A \in \mathcal{A} \,|\, \omega(A^*A) = 0\}.$$

We want to show that $\mathscr{I}^\omega$ is a left ideal in $\mathcal{A}$. First, $\mathscr{I}^\omega$ is linear for the following reason.

(a) For $A \in \mathscr{I}^\omega$, $\lambda \in \mathbb{C}$ now

$$\omega((\lambda A)^*(\lambda A)) = |\lambda|^2 \omega(A^*A)$$

implies $\lambda A \in \mathscr{I}^\omega$.

(b) For $A, B \in \mathscr{I}^\omega$ we have

$$\omega((A+B)^*(A+B)) = \underbrace{\omega(A^*A)}_{=0} + \underbrace{\omega(B^*B)}_{=0} + 2\,\mathrm{Re}\,\omega(A^*B) = 0.$$

It is easy to see that $2\,\mathrm{Re}\,\omega(A^*B)$ vanishes by Cauchy-Schwarz.
Now $A \in \mathscr{I}^\omega$ implies $|\langle A, B\rangle|^2 \leq \omega(A^*A)\omega(B^*B)$ so $\langle A, B\rangle = 0$ for all $B \in \mathcal{A}$ and thus

$$\mathscr{I}^\omega = \langle A \in \mathcal{A} \,|\, \omega(AB) = 0 \;\forall_{B\in\mathcal{A}}\}.$$

Finally, $\mathscr{I}^\omega$ is a left ideal because

$$\omega((BA)^*C) = \omega(A^*(B^*C)) = 0 \implies BA \in \mathscr{I}^\omega$$

for any $A \in \mathscr{I}^\omega$, $B, C \in \mathcal{A}$. Now we define $D_\omega = \mathcal{A}\backslash\mathscr{I}^\omega$ and a scalar product $\langle [A], [B]\rangle = \langle A, B\rangle$ which is well-defined as can be verified readily. Our Hilbert space $\mathcal{H}$ then is given by the completion of D_ω, which by construction is dense in $\mathcal{H}$. For $A, B \in \mathcal{A}$, $I \in \mathscr{I}^\omega$ we have

$$A(B+I) = AB + \underbrace{AI}_{\in\mathscr{I}^\omega} \implies A(B+I) \in [AB].$$

Thus we can define π_ω by

$$\pi_\omega(A)[B] = [AB],$$

as well as $\Omega_\omega = [\mathbb{1}]$. Then $\pi_\omega(A)\Omega = [A]$ and $\{\pi_\omega(A)\Omega \,|\, A \in \mathcal{A}\} = D_\omega \subset \mathcal{H}$ is dense. Finally, we have to check (1.4).

$$\langle \Omega_\omega, \pi_\omega(A), \Omega_\omega\rangle = \langle [\pi], \pi_\omega(A)[\pi]\rangle = \langle [\pi], [A\pi]\rangle = \langle [\mathbb{1}], [A]\rangle = \omega(\mathbb{1}A) = \omega(A)$$

Now for the uniqueness. Let $(\tilde{\mathcal{H}}, \tilde{D}, \tilde{\pi}, \tilde{\Omega})$ be another cyclic representation which satisfies (1.4) with $\tilde{D} = \tilde{D}_0 = \{\tilde{\pi}(A)\tilde{\Omega} \,|\, A \in \mathcal{A}\}$. Define

$$U : D_\omega \to \tilde{D} \qquad U\pi_\omega(A)\Omega_\omega = \tilde{\pi}(A)\tilde{\Omega}$$

After showing that U is well-defined and unitary, one extends it to $\mathcal{H}$ and sees that $UD_\omega = \tilde{D}$ and $U\pi_\omega(A)U^* = \tilde{\pi}(A)$. □

Proposition 1.37. *Consider finite direct sums*

$$\mathcal{A} = \mathbb{C} \oplus \mathscr{S}(\mathbb{R}^n) \oplus \mathscr{S}(\mathbb{R}^n \times \mathbb{R}^n) \oplus \ldots$$

with seminorms $\|f\|^{(m)}_{\alpha\beta} = \|f_m\|_{\alpha\beta}$, *where*

$$f = f_1 \oplus \ldots \oplus f_m \oplus \ldots \oplus f_n$$

and $f_n \in \mathscr{S}(\mathbb{R}^{n\cdot m})$, and consider the product

$$\begin{aligned} fg &= [f^{(0)} \oplus f_1 \oplus \ldots \oplus f_n] \otimes [g^{(0)} \oplus \ldots \oplus g^{(k)}] \\ &= f^{(0)} \otimes g^{(0)} \oplus (f^{(0)} \otimes g^{(1)} + f_1 \otimes g^{(0)}) \oplus (f^{(0)} \otimes g^{(2)} + f_1 \otimes g^{(1)} + f_2 \otimes g^{(0)}) \oplus \ldots, \end{aligned}$$

as well as the $*$-operation

$$(f^*)^{(m)}(x_1, \ldots, x_m) = \overline{f_m(x_m, \ldots, x_1)}.$$

Then $\mathcal{A}$ is locally convex space and $*$-algebra. Further, the operations are continuous so $\mathcal{A}$ is a topological $*$-algebra with unit $\mathbb{1} = 1 \oplus 0 \oplus 0 \oplus \ldots \oplus 0$.

Definition 1.38. *The $\mathcal{A}$ just defined is called Borchers-Uhlmann-algebra (BU-Algebra).*

The following proposition is a straightforward consequence of the definitions.

Proposition 1.39. *Let $(\mathcal{H}, D, \Phi, \Omega)$ be a hermitian quantum field with Wightman distributions $W^{(m)}$. The functional*

$$\omega(f^{(0)} \oplus f_1 \oplus \ldots \oplus f_m) = \sum_{\alpha=0}^{m} W^{(\alpha)} f^{(\alpha)} \tag{1.5}$$

is a state of the BU-Algebra.

Proposition 1.40. *Let $(\mathcal{H}, D, \Phi, \Omega)$ be a hermitian quantum field. There is a unique cyclic representation $(\mathcal{H}, D, \pi, \Omega)$ such that*

$$\pi(f_1 \otimes \ldots \otimes f_m) = \Phi(f_1) \ldots \Phi(f_m).$$

Further, π is the GNS representation of ω from (1.5).

Proof idea. First, we define $\mathcal{A}_0 \subset \mathcal{A}$ generated by tensor products $f_1 \otimes \ldots \otimes f_m$ which again is a $*$-algebra. The representation π is the GNS representation of ω restricted to $\mathcal{A}_0$ by the definition of the Wightman distributions. Consider the GNS representation $(\mathcal{H}_\omega, D_\omega, \pi_\omega, \Omega_\omega)$ of $\mathcal{A}$ with respect to the state ω. If we restrict π_ω to $\mathcal{A}_0$ we get a new representation $\tilde{\pi}$. We show that Ω_ω is cyclic for $\tilde{\pi}$ by using the following facts:

(a) The span of tensor products $f_1 \otimes \ldots \otimes f_m$ is dense in $\mathscr{S}(\mathbb{R}^n \times \ldots \times \mathbb{R}^n)$ by the N representation theorem; cf. [RS80] Theorem V.13. Hence $\mathcal{A}_0$ is dense in $\mathcal{A}$.

(b) The representation is continuous in the sense that

$$\mathscr{S}(\mathbb{R}^n \times \ldots \times \mathbb{R}^n) \ni f \longmapsto \langle \xi, \pi_\omega(f)\psi \rangle$$

is in $\mathscr{S}'(\mathbb{R}^{n\cdot m})$ (by the nuclear theorem) for all $\xi, \psi \in D_\omega$.

Assume that Ω_ω is not cyclic for $\tilde{\pi}$. Then there is a $0 \neq \xi \in \mathcal{H}_\omega$ such that $\langle \tilde{\pi}(f)\Omega_\omega, \xi \rangle = 0$ for all $f \in \mathcal{A}_0$. But since π_ω is the GNS representation of $\mathcal{A}$ with respect to ω, the vector Ω_ω is cyclic for π_ω, i.e. there is a $f \in \mathcal{A}$ such that $\langle \pi_\omega(f)\Omega_\omega, \xi \rangle \neq 0$. By the denseness

of $\mathcal{A}_0$ in $\mathcal{A}$ there is a sequence of $f_k \in \mathcal{A}_0$ $k \in \mathbb{N}$ converging to $f \in \mathcal{A}$. Hence by the continuity just stated we get

$$\lim_{k\to 0}\langle\tilde{\pi}(f_k)\Omega_\omega, \xi\rangle = \lim_{k\to 0}\langle\pi_\omega(f_k)\Omega_\omega, \xi\rangle = \langle\pi_\omega(f)\Omega_\omega, \xi\rangle$$

in contradiction to $\langle\tilde{\pi}(f_k)\Omega_\omega, \xi\rangle = 0$ and $\langle\pi_\omega(f)\Omega_\omega, \xi\rangle \neq 0$. Hence Ω_ω is a cyclic vector for $\tilde{\pi}$ and the latter is therefore unitarily equivalent to the GNS representation of $\mathcal{A}_0$ with respect to ω. Hence there exists a unitary $U : \mathcal{H}_\omega \to \mathcal{H}$ such that

$$U\pi_\omega(f_1)\dots\pi_\omega(f_m)U^* = \Phi(f_1)\dots\Phi(f_m).$$

For arbitrary $f \in \mathscr{S}(\mathbb{R}^{n\cdot m})$ we define $\Phi(f) := U\pi_\omega(f)U^*$, which is the extension we are looking for. The uniqueness follows from the fact that the matrix elements $\langle\xi, \Phi(f)\psi\rangle$ for $\xi, \psi \in D_0$ are uniquely determined by ω. □

Remark 1.41 (Fields and representations)**.** The previous results show that a quantum field (H, D_0, Φ, Ω) can be recovered (up to unitary equivalence) from its Wightman distributions $W^{(m)}$. The $W^{(m)}$ define a state ω on the BU-algebra $\mathcal{A}$, this state has a GNS representation $(\mathcal{H}_\omega, D_\omega, \pi_\omega, \Omega_\omega)$ which by Proposition 1.39 is unitarily equivalent to the representation defined by Φ, i.e. there is a unitary U with $U\Phi(f_1)\cdots\Phi(f_m)U^* = \pi_\omega(f_1\otimes\cdots\otimes f_m)$ for all $f_1, \dots, f_m \in \mathscr{S}(\mathbb{R}^n)$ and all $m \in \mathbb{N}$. Hence we recover a unitarily equivalent copy of Φ by

$$\mathscr{S}(\mathbb{R}^n) \ni f \mapsto \Phi_\omega(f) = \pi_\omega(f) \in \mathcal{L}(D_\omega, D_\omega). \tag{1.6}$$

The last equation shows that we even have a little bit more. If ω is any continuous state of $\mathcal{A}$, it defines its GNS representation $(\mathcal{H}_\omega, D_\omega, \pi_\omega, \Omega_\omega)$ and via Eq. (1.6) a map Φ_ω from $\mathscr{S}(\mathbb{R}^n)$ to $\mathcal{L}(D_\omega, D_\omega)$. Hence the 4-tuple $(\mathcal{H}_\omega, D_\omega, \Phi_\omega, \Omega_\omega)$ is a quantum field, and would be a hermitian quantum field if the $\Phi_\omega(f)$ would be essentially self-adjoint for real-valued f. This is unfortunately something we can not get for free. Hence it is – at least in the present context – advantageous to drop the self-adjointness condition from Def. 1.22. In that case we get two one-to-one correspondences:

$$\text{Hermitian quantum field} \longleftrightarrow \text{Representations of } \mathcal{A}$$

and

$$\text{Sequences of Wightman distributions} \longleftrightarrow \text{States of } \mathcal{A}.$$

Let us come back to Wightman fields. The next proposition tells us how the additional properties translate into properties of the Wightman distributions.

Proposition 1.42 ([SW64], Chapter 3.3)**.** *Let $(\mathcal{H}, D, \Phi, \Omega, U)$ a Wightman quantum field. Then the following statements hold.*

(a) All the $W^{(m)}$ are Poincaré-invariant, so

$$W^{(m)}((a, \Lambda)f) = W^{(m)}(f)$$

where $((a,\Lambda)f) = f(\Lambda^{-1}(x_1 - a), \ldots)$.

(b) *(Spectrum condition). The Fourier transform* $\hat{W}^{(m)}$ *of* $W^{(m)}$ *has support in the set of* $p = (p_1, \ldots, p_m) \in \mathbb{R}^{n \cdot m}$ *with* $\sum_{j=1}^{m} p_j = 0$ *and* $\sum_{j=k}^{m} p_j \in V^+$ *for any* $k = 2, \ldots, m$.

(c) *(Locality). Whenever* x_k *and* x_{k+1} *are spacelike separated, we have*

$$W^{(m)}(x_1, \ldots, x_k, x_{k+1}, \ldots, x_m) = W^{(m)}(x_1, \ldots, x_{k+1}, x_k, \ldots, x_m).$$

(d) *(Cluster property).*

$$\lim_{\|a\| \to \infty} W^{(m)}(x_1, \ldots, x_k, x_{k+1} + a, \ldots, x_m + a) = W^{(k)}(x_1, \ldots, x_k) W^{(m-k)}(x_{k+1}, \ldots, x_m)$$

Remark 1.43 (Interpretation)**.** The $W^{(m)}$ can be regarded as correlation functions. E.g. $W^{(2)}$ – which is often called the 3-point function – describes the correlations between two field operators $\Phi(f)$, $\Phi(g)$ in the vacuum. If $W^{(2)}(f \otimes g)$ factorizes, measurements of $\Phi(f)$ and $\Phi(g)$ in the vaccuum are uncorrelated. By the cluster property this happens if the supports of f and g are very far apart. Spacial separation is, on the other hand, not sufficient. We only get invariance under permutations of f and g. A closer analysis shows that the correlations of the fields between spacelike separated events in the vaccum representation can be arbitrarily high, if the distance between the events is small enough. Poincaré invariance shows that these correlations look always the same in all inertial frames. In particular translations are interesting since it allows us to rewrite $W^{(2)}$ in terms of a distribution $W_2 \in \mathscr{S}(\mathbb{R}^4)$ such that we formally get

$$W^{(2)}(f \otimes g) = \int_{\mathbb{R}^4} \int_{\mathbb{R}^4} W^{(2)}(x,y) f(x) g(y)\, dx\, dy = \int_{\mathbb{R}^4} \int_{\mathbb{R}^4} W_2(x-y) f(x) g(y)\, dx\, dy.$$

Similarly we can rewrite $W^{(m)}$ in terms of a distribution W_m in $\mathscr{S}(\mathbb{R}^{(m-1)\cdot 4})$. The W_m have interesting analyticity properties which can be used to prove structural results like the PCT theorem The spectrum condition translates to support properties in momentum space.

The last result in this chapter combines the construction of a quantum field from a sequence of Wightman distributions discussed above with discussion from the last proposition. In a nutshell it says that if the $W^{(m)}$ have all the properties from Proposition 1.42, the reconstructed field is a Wightman QF.

Theorem 1.44 ([SW64], Theorem 3.7)**.** *Consider* $\mathbb{R}^n = \mathbb{R}^4$. *Let a state* $\omega : \mathcal{A} \to \mathbb{C}$ *of the BU-Algebra* $\mathcal{A}$ *be given and consider the corresponding GNS representation* $(\mathcal{H}_\omega, D_\omega, \pi_\omega, \Omega_\omega)$. *Further let* $\omega(f) = \sum_{\alpha=0}^{m} W^{(\alpha)}(f^{(\alpha)})$ *for* $W^{(\alpha)} \in \mathscr{S}'(\mathbb{R}^{4\alpha})$ *and let* $W^{(m)}$ *satisfy the four conditions from Proposition 1.42. Then there exists a strongly continuous unitary representation* $U_\omega : \mathcal{P}_+^\uparrow \to \mathcal{U}(\mathcal{H}_\omega)$ *such that* $(\mathcal{H}_\omega, D_\omega, \pi_\omega, \Omega_\omega, U_\omega)$ *is a Wightman quantum field, although without the self-adjointness of the field operators.*

2 The Free Scalar Field

2.1 Representation of the Poincaré Group

Definition 2.1. *Let a group G, a $\mathbb{C}$-vector space V and $\xi : G \times G \to \mathbb{R}$ be given. A map $U : G \to \mathrm{GL}(V)$ is called projective representation of G if*

$$U(f)U(g) = e^{i\xi(f,g)}U(fg).$$

The exponents ξ can not be chosen freely, but have to satisfy some contraints, which can be easily deduced (e.g. look at $U(f)U(g)U(h)$). If all of them are satisfied, ξ is called a cocycle.

By Wigner, a free, relativistic elementary particle is described by a projective unitary representation of $\mathcal{P}_+^\uparrow$ which actually has to be irreducible emphasizing on the term *elementary*. So Wigners goal was to classify all the projective representations of $\mathcal{P}_+^\uparrow$. For more on this topic, we refer to [Bog+90, Chapter 7.2] and [RS75, Chapter IX.8]. The representations of $\mathcal{P}_+^\uparrow$ can be divided into classes via the mass, given by $m^2 \in \mathbb{R}$. Here the physical case obviously is $m^2 \geq 0$. All representations with $m^2 > 0$ are further characterized, up to unitary equivalence, by the spin $s \in \frac{1}{2}\mathbb{N}_0 = \{0, \frac{1}{2}, 1, \frac{3}{2}, \ldots\}$ of the representation. There exists a unique unitary projective representation of $\mathcal{P}_+^\uparrow$ with $m^2 > 0$, $s \in \frac{1}{2}\mathbb{N}_0$.

We will proceed with the easiest case $m^2 > 0$, $s = 0$. The first step is to define the "mass shell"

$$H_m = \{p \in \mathbb{R}^4 \,|\, \eta(p,p) = m^2, p^0 > 0\}. \tag{2.1}$$

Note, that H_m is an orbit of $\mathrm{SO}^\uparrow(1,3)$ since for $\Lambda \in \mathrm{SO}^\uparrow(1,3)$, $p \in H_m$ we have $\Lambda p \in H_m$.

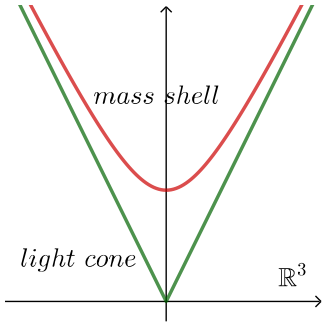


Figure 4: The mass shell from Eq. (2.1) as part of the forward light cone.

We parametrize H_m by

$$\mathbb{R}^3 \ni p \longmapsto j(p) = (\omega(p), p) \in H_m \tag{2.2}$$

with $\omega(p) = \sqrt{\|p\|^2 + m^2}$. This j can be regarded as a coordinate system such that H_m becomes a smooth manifold. An only slightly more detailed analysis shows that it actually is a smooth submanifold of $\mathbb{R}^4$. This shows in paticular that H_m is a locally

compact topological sapce which is homeomorphic to $\mathbb{R}^3$. Hence we can consider Borel subsets of H_m and define the following Lorentz invariant measure.

Proposition 2.2 ([RS75], Thm IX.37)**.** *For all Borel subsets $\Omega \subset H_m$ we write*

$$\Omega_m(\Omega) = \int_{j^{-1}(\Omega)} \frac{dp}{\omega(p)}.$$

This defines a Borel measure on H_m which is Lorentz invariant: $\Omega_m(\Lambda\Omega) = \Omega_m(\Omega)$ for all $\Lambda \in \mathrm{SO}^\uparrow(1,3)$ and all Borel sets $\Omega \subset H_m$.

With Ω_m we can define the Hilbert space $\mathrm{L}^2(H_m, \Omega_m)$ of square-integrable functions on H_m. Since Ω_m is Lorentz invariant, a transformation $f(\,\cdot\,) \mapsto f(\Lambda^{-1}\,\cdot\,)$ with a Lorenz transformation Λ leads to a unitary operator $U(\Lambda)$ on $\mathrm{L}^2(H_m, \Omega_m)$, and therefore to a unitary representation of $\mathrm{SO}^\uparrow(1,3)$ which is easily shown to be strongly continuous, i.e. the map

$$\mathrm{SO}^\uparrow(1,3) \ni \Lambda \mapsto U(\Lambda)\psi \in \mathrm{L}^2(H_m, \Omega_m) \tag{2.3}$$

is continuous for all $\psi \in \mathrm{L}^2(H_m, \Omega_m)$. Combining this with a representation of the translation group leads to a representation of $\mathcal{P}_+^\uparrow$.

Proposition 2.3. *The map $\mathcal{P}_+^\uparrow \ni (b, \Lambda) \mapsto U(b, \Lambda) \in \mathcal{U}\big(\mathrm{L}^2(H_m, \Omega_m)\big)$ with*

$$(U(b,\Lambda)\psi)(p) = e^{i\eta(b,p)}\phi(\Lambda^{-1}p).$$

is a strongly continuous, irreducible, unitary representation of the restricted Poincaré group.

Proof. The proof is easy and therefore left as an exercise. Note that strong coninuity can be defined as in Eq. (2.3). □

Remark 2.4 (Interpretation)**.** The representation just constructed contains (almost) everything we need for the quantum mechanics of one relativistic particle of mass m and spin 0. Due to strong continuity we can define the generators of the translations as self-adjoint operators P_j, $j = 0, \ldots, 3$

$$P_j\psi = -i\,\frac{d}{d\lambda} e^{i\eta(\lambda e_j, p)}\psi(p)\bigg|_{\lambda=0}$$

where the domains $D(P_j)$ are consisting of exactly those ψ for which the given limit exists; cf. Remark 1.25. Explicitly we get

$$(P_0\psi)(\omega(k),k) = \omega(k)\psi(\omega(k),k), \quad (P_j\psi)(\omega(k),k) = -k_j\psi(\omega(k),k)\ j = 1,2,3.$$

The P_j for $j = 1, 2, 3$ describe the three-momentum of the particle, P_0 is its Hamiltonian. With the reasoning already pointed out in Remark 1.25 the operator $P_b = \sum_j b_j P_j$ describes the Hamiltoian in the inertial frame moving with four-velocity $b \in H_1$. Hence,

we have almost everything we need for a quantum mechanical description. The only missing component are the position operators. They do not come out of the representation U directly, but they have to satisfiy a number of conditions. The corresponding analysis was carried out by Newton and Wigner in 1949. The resulting theory has serious locality problems as already mentioned in the introduction. We omit the discussion at that point, but refer the reader to original work.

We can use the parametrization map j to pull everything back to the Hilbert space $\mathrm{L}^2(\mathbb{R}^3)$. This is sometimes useful – in particular if we want to compare results with expressions known from the physics literature.

Proposition 2.5. *The map* $J : \mathrm{L}^2(H_m, \Omega_m) \to \mathrm{L}^2(\mathbb{R}^3)$ *given by*

$$(J\psi)(k) = \frac{(\psi \circ j)(k)}{\sqrt{\omega(k)}} = \frac{\psi(\omega(k), k)}{\sqrt{\omega(k)}}$$

is a unitary operator.

Proof. This is easily checked and therefore left as an exercise. □

2.2 The Klein-Gordon Equation

Our goal is to quantize the Klein-Gordon equation. Therefore, we have to have a closer look on the classical solutions and their relations to the discussion of the previous section. Hence, let us start with

$$(\Box + m^2)\Psi := \frac{\partial^2 \Psi}{\partial t^2} - \Delta\Psi + m^2\Psi = 0 \tag{2.4}$$

and

$$\Psi \in \mathcal{K}_{\mathbb{C}} := \{\Psi \in C^2(\mathbb{R}^4, \mathbb{C}) \,|\, \Psi_t(\,\cdot\,) := \Psi(t, \,\cdot\,) \in \mathscr{S}(\mathbb{R}^3)\}. \tag{2.5}$$

This somewhat strange function space arises from the desire to use the Fourier transform with respect to the position variable $x \in \mathbb{R}^3$. We are interested in solutions of the Cauchy problem with initial data in $\mathscr{S}(\mathbb{R}^3)$:

$$\Psi(0, x) = f(x) \quad \text{and} \quad \partial_t \Psi(0, x) = p(x) \quad \text{with} \quad f, p \in \mathscr{S}(\mathbb{R}^3).$$

Using the shortcut notation $\Psi_t(x) := \Psi(t, x)$ as in Eq. (2.5), now $\hat{\Psi}_t$ is the Fourier transform with respect to $x \in \mathbb{R}^3$. Fourier transforming all of (2.4) (in $\mathbb{R}^3$) gives

$$\frac{\partial^2 \hat{\Psi}_t(k)}{\partial t^2} + \|k\|^2 \hat{\Psi}_t(k) + m^2 \hat{\Psi}_t(k) = 0.$$

So for any k, we get an ordinary differential equation of second order in t which yields the solution

$$\hat{\Psi}_t(k) = b(k) e^{i\omega(k)t} + c(k) e^{-i\omega(k)t}$$

with $\omega(k) = \sqrt{\|k\|^2 + m^2} > 0$ for all k since $m > 0$. Using the initial conditions, we easily get

$$\hat{f}(k) = b(k) + c(k) \qquad \hat{p}(k) = i\omega(k)(b(k) - c(k))$$

and thus

$$b(k) = \frac{1}{2}\Big(\hat{f}(k) - \frac{i}{\omega(k)}\hat{p}(k)\Big) \qquad c(k) = \frac{1}{2}\Big(\hat{f}(k) + \frac{i}{\omega(k)}\hat{p}(k)\Big). \tag{2.6}$$

Proposition 2.6. *For all initial data $f, p \in \mathscr{S}(\mathbb{R}^3)$ there exists a unique solution $\Psi \in \mathcal{K}_{\mathbb{C}}$ to the Klein-Gordon equation which is given by*

$$\Psi(t,x) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} \Big(b(k)e^{i(k\cdot x + \omega(k)t)} + c(k)e^{i(k\cdot x - \omega(k)t)}\Big)\, d^3k \tag{2.7}$$

for all $t \in \mathbb{R}$. The functions $b(k), c(k)$ are derived from f, p as in (2.6) and $\omega(k) = \sqrt{\|k\|^2 + m^2}$.

Proof. We haven't shown yet that the presented solution really is in the set $\mathcal{K}_{\mathbb{C}}$. To this end first note that ω and $1/\omega$ are smooth functions. Furthermore they are polynomially bounded, and the same is true for all its powers. Hence, for all $f \in \mathscr{S}(\mathbb{R}^3)$ the products $\omega^n f$ and $\omega^{-n} f$ are Schwartz functions again. The same is true if we multiply for fixed $t \in \mathbb{R}$ with a phase factor $e^{\pm it\omega(k)}$. With this knowledge we conclude from Eq. (2.6) that b and c are in $\mathscr{S}(\mathbb{R}^3)$. Hence the expression in Eq. (2.7) is for fixed t the sum of the inverse Fourier transform of two Schwartz functions, which is again a Schwartz function in variable x for fixed t. □

Remark 2.7. For later use let us recall that we can multiply and divide Schwartz functions by ω and still get Schwartz functions. This follows since $\omega(k)$ is bounded from below by $m > 0$ (hence we avoid the singularities at zero) and polynomially bounded from above (the same is obviously true for all powers of ω). We have already used such an argument in Example 1.8 to define the product of a distribution and a polynomially bounded function.

Now we want to look for a real-valued solution Ψ, $f \in \mathscr{S}(\mathbb{R}^3, \mathbb{R})$. For the Fourier transform this means

$$\overline{\hat{f}(k)} = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} f(x) e^{ik\cdot x}\, dx,$$

so $\overline{\hat{f}(k)} = \hat{f}(-k)$ which implies $b(-k) = \overline{c(k)}$. Putting this into the solution we have by Proposition 2.6 gives us

$$\Psi(t,x) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} \overline{c}(-k) e^{i(k\cdot x + \omega(k)t)}\, dk + \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} c(k) e^{i(k\cdot x - \omega(k)t)}\, dk.$$

In the first integral, we substitute $k \to -k$, which produces no sign, so putting in $a(k) = \sqrt{2\omega(k)}c(k)$ gives the real-valued solution

$$\Psi(t,x) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} \left(\bar{a}(k) e^{-i(k\cdot x - \omega(k)t)} + a(k) e^{i(k\cdot x - \omega(k)t)} \right) \frac{dk}{\sqrt{2\omega(k)}}. \tag{2.8}$$

Obviously the argument of the integral is real as sum of something and its complex conjugate. Now let us introduce $A \in \mathrm{L}^2(H_m, \Omega_m)$ by $A = J^{-1}a$, where J is unitary from Proposition 2.5. We can express a in terms of A by

$$a(k) = (JA)(k) = \frac{A(\omega(k), k)}{\sqrt{\omega(k)}}.$$

Inserting this into Eq. (2.8) and using the abbreviation $\xi = (t, x)$, we get an integral over the mass shell.

$$\Psi(\xi) = \frac{1}{(2\pi)^{3/2}} \int_{H_m} \left(\overline{A}(\lambda) e^{i\eta(\xi,\lambda)} + A(\lambda) e^{-i\eta(\xi,\lambda)} \right) \frac{d\Omega(\lambda)}{\sqrt{2}}.$$

We summarize this discussion in the following two propositions.

Proposition 2.8. *Consider the sets $\mathcal{M} = \mathscr{S}(\mathbb{R}^3) \times \mathscr{S}(\mathbb{R}^3)$ and $\mathscr{S}(H_m) = J^{-1}\big(\mathscr{S}(\mathbb{R}^3)\big) \subset \mathrm{L}^2(H_m, \Omega_m)$. For each $(f,p) \in \mathcal{M}$ the function $A_{f,p} : H_m \to \mathbb{C}$ with*

$$A_{f,p}(\omega(k), k) = \frac{1}{\sqrt{2}} \Big(\omega(k) \hat{f}(k) + i\hat{p}(k) \Big)$$

is in $\mathscr{S}(H_m)$. The corresponding map $\mathcal{M} \ni (f,p) \mapsto A_{f,p} \in \mathscr{S}(H_m)$ is real linear and invertible. The inverse is given by $\mathscr{S}(H_m) \ni A \mapsto (f_A, p_A) \in \mathcal{M}$ with

$$\begin{aligned} f_A(x) &= \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} \left(a(k) e^{ik\cdot x} + \bar{a}(k) e^{-ik\cdot x} \right) \frac{dk}{\sqrt{2\omega(k)}} \\ p_A(x) &= \frac{i}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} \left(a(k) e^{ik\cdot x} - \bar{a}(k) e^{-ik\cdot x} \right) \frac{\sqrt{\omega(k)}}{\sqrt{2}} dk \end{aligned} \tag{2.9}$$

with $a = JA$ and the unitary J from Proposition 2.5.

Proof. We have to show that $JA_{f,p}$ with $(f,p) \in \mathcal{M}$ and

$$JA_{f,p}(k) = \frac{1}{\sqrt{2}} \left(\sqrt{\omega(k)} \hat{f}(k) + \frac{i}{\sqrt{\omega(k)}} \hat{p}(k) \right)$$

is in $\mathscr{S}(\mathbb{R}^3)$. But this follows from the fact that f, p are Schwartz functions and Remark 2.7 above. Linearity of the map is obvious, and invertibility follows from the existence of an inverse. That the given map is really the inverse is left as an exercise to the reader (this can be easily done with slight modifications of the calculations above). □

Proposition 2.9. *For each* $A \in \mathscr{S}(H_m) = J^{-1}\big(\mathscr{S}(\mathbb{R}^3)\big) \subset \mathrm{L}^2(H_m, \Omega_m)$ *the expresssion*

$$\Psi = \sqrt{2}\,\mathrm{Re}\,\mathcal{F}\overline{A} \quad \textit{with} \quad (\mathcal{F}\overline{A})(\xi) = \frac{1}{(2\pi)^{3/2}} \int_{H_m} e^{i\eta(\xi,\lambda)}\,\overline{A}(\lambda) d\Omega(\lambda)$$

hence

$$\Psi(\xi) = \frac{1}{(2\pi)^{3/2}} \int_{H_m} \Big(\overline{A}(\lambda) e^{i\eta(\xi,\lambda)} + A(\lambda) e^{-i\eta(\xi,\lambda)}\Big) \frac{d\Omega(\lambda)}{\sqrt{2}}. \tag{2.10}$$

is a solution of the Klein-Gordon equation (2.4), and an element of the space $\mathcal{K} = \{f \in \mathcal{K}_{\mathbb{C}} \,|\, f = \overline{f}\}$ *with* $\mathcal{K}_{\mathbb{C}}$ *from Eq. (2.5). The corresponding initial data are given as* $(f, p) = (f_A, p_A)$ *with the expressions from Proposition 2.8. Similarly, we get the solution with initial data* $(f, p) \in \mathcal{M}$ *if* $A = A_{f,p}$ *holds.*

Proof. That $\mathcal{F}\overline{A}$ is a solution follows easily from

$$\frac{\partial^2}{\partial t^2}\mathcal{F}\overline{A}(t,x) = -\frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} e^{i(\omega(k)t - k\cdot x)} \omega(k)^2 \bar{a}(k) \frac{dk}{\sqrt{\omega(k)}}$$
$$(\Delta\mathcal{F}\overline{A})(t,x) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} e^{i(\omega(k)t - k\cdot x)} \|k\|^2 \bar{a}(k) \frac{dk}{\sqrt{\omega(k)}}.$$

With $a = JA$ and the unitary J from Proposition 2.5. Hence $\mathrm{Re}\,\mathcal{F}\overline{A}$ is a real-valued solution. The initial data for the solution $\mathcal{F}\overline{A}$ are easily calculated as

$$(\mathcal{F}\overline{A})(0,x) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} e^{-ix\cdot k} \frac{\bar{a}(k)}{\sqrt{\omega(k)}}\, dk, \tag{2.11}$$

$$\Big(\frac{\partial}{\partial t}\mathcal{F}\overline{A}\Big)(0,x) = \frac{i}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} e^{-ix\cdot k} \bar{a}(k) \sqrt{\omega(k)}\, dk \tag{2.12}$$

and are obviously Schwartz functions (cf. Remark 2.7). Hence by Proposition 2.6 the solution $\mathcal{F}\overline{A}$ is in $\mathcal{K}_{\mathbb{C}}$, its real part, obviously, is in $\mathcal{K}$. The remaining statements follow immediately from Eqs. (2.11), (2.12) and Proposition 2.8. □

Remark 2.10 (Fourier transform)**.** In the last proposition we have implicitly introduced the map

$$\mathcal{F} : \mathscr{S}(H_m) \to C^\infty(\mathbb{R}^4),\ A \mapsto \mathcal{F}A \quad \text{with} \quad \mathcal{F}A(\xi) = \int_{H_m} e^{i\eta(\xi,\lambda)} g(\lambda)\, d\Omega(\lambda)$$

and the domain

$$\mathscr{S}(H_m) = J^{-1}\big(\mathscr{S}(\mathbb{R}^3)\big) \subset \mathrm{L}^2(H_m, \Omega_m).$$

Since J is unitary, $\mathscr{S}$ is an isomorphic copy of $\mathscr{S}(\mathbb{R}^3)$. Hence we can equip it with the same topology such that it becomes a Fréchet space. This implies in particular that we can define the topological dual $\mathscr{S}'(H_m)$ and embed $\mathscr{S}(H_m)$ into $\mathscr{S}'(H_m)$ via a map

$\mathscr{S}(H_m) \ni A \mapsto \phi_A \in \mathscr{S}'(H_m)$ with

$$\phi_A(f) = \int_{H_m} A(\lambda) f(\lambda)\, d\Omega_m(\lambda).$$

All our knowledge about tempered distributions can be translated to the "mass shell supported" versions in $\mathscr{S}(H_m)$ via the map J.

The map $\mathcal{F}$ can be regarded as some form of Fourier transform and as such it can be extended to the distribution space $\mathscr{S}'(H_m)$. This can be done in terms of

$$E : \mathscr{S}(\mathbb{R}^4) \ni f \mapsto \sqrt{2\pi}\tilde{f}\big|_{H_m} \quad \text{with} \quad \tilde{f}(\lambda) = \frac{1}{2\pi}\int_{\mathbb{R}^4} e^{i\eta(\xi,\lambda)} f(\xi)\, d\xi. \tag{2.13}$$

Note that we have defined the Fourier transform $\tilde{f}$ for this particular purpose with the Minkowski metric η rather than the usual scalar product. With the map E we write

$$\begin{aligned}
\phi_A(Ef) &= \sqrt{2\pi}\int_{H_m} \frac{1}{(2\pi)^2}\int_{\mathbb{R}^4} e^{i\eta(\xi,\lambda)} f(\xi)\, d\xi\, A(\lambda)\, d\Omega(\lambda) \\
&= \int_{\mathbb{R}^4} \left[\frac{1}{(2\pi)^{3/2}}\int_{H_m} e^{i\eta(\xi,\lambda)} A(\lambda)\, d\Omega_m(\lambda)\right] f(\xi)\, d\xi \\
&= \int_{\mathbb{R}^4} \mathcal{F}A(\xi) f(\xi)\, d\xi = \phi_{\mathcal{F}A}(f)
\end{aligned}$$

with $\phi_{\mathcal{F}A} \in \mathscr{S}'(\mathbb{R}^4)$ the regular distribution belonging to $\mathcal{F}A \in C^\infty(\mathbb{R}^4)$ – please check that this is well-defined although $\mathcal{F}A$ is not in L^p, cf. Example 1.6. Summarizing this calculation we can conclude that $\mathscr{S}'(H_m) \ni \phi \mapsto \phi \circ E \in \mathscr{S}'(\mathbb{R}^4)$ is the (unique) weakly contiunous extension of $\mathcal{F}$ to $\mathscr{S}'(H_m)$; cf. the discussion in Remark 1.7. The following proposition shows how to recreate the real solution $\operatorname{Re}\mathcal{F}$ in terms of the map E.

Proposition 2.11. *For each $A \in \mathscr{S}(H_m)$ the complex linear extension of the map*

$$\mathscr{S}(\mathbb{R}^4, \mathbb{R}) \ni f \mapsto \sqrt{2}\operatorname{Re}\phi_{\overline{A}}(Ef) = \sqrt{2}\operatorname{Re}\int_{H_m} Ef(\lambda)\overline{A}(\lambda) d\Omega(\lambda)$$

coincides with the regular distribution $\phi_{\sqrt{2}\operatorname{Re}\mathcal{F}\overline{A}}$

$$\phi_{\sqrt{2}\operatorname{Re}\mathcal{F}\overline{A}}(f) = \int_{\mathbb{R}^4} \sqrt{2}\operatorname{Re}\mathcal{F}\overline{A}(\xi) f(\xi) d\xi$$

associated to the solution $\sqrt{2}\operatorname{Re}\mathcal{F}\overline{A}$ of the Klein-Gordon equation.

Proof. We calculate $\phi_{\overline{A}}(Ef)$ with real-valued test function f:

$$\begin{aligned}\sqrt{2}\,\mathrm{Re}\,\phi_{\overline{A}}(Ef) &= \frac{1}{\sqrt{2}}\left(\phi_{\overline{A}}(Ef) + \overline{\phi_{\overline{A}}(Ef)}\right)\\ &= \int_{H_m} \overline{A}(\lambda)\frac{1}{(2\pi)^{3/2}}\int_{\mathbb{R}^4} e^{i\eta(\lambda,\xi)} f(\xi)\,d\xi\,d\Omega(\lambda) + \\ &\qquad \int_{H_m} A(\lambda)\frac{1}{(2\pi)^{3/2}}\int_{\mathbb{R}^4} e^{-i\eta(\lambda,\xi)} f(\xi)\,d\xi\,d\Omega(\lambda)\end{aligned}$$

Hence

$$\sqrt{2}\,\mathrm{Re}\,\phi_{\overline{A}}(Ef) = \int_{\mathbb{R}^4} f(\xi)\sqrt{2}\,\mathrm{Re}(\mathcal{F}\overline{A})(\xi)\,d\xi,$$

as claimed. □

It is left as an Exercise to the reader that we can use this strategy to generate weak (i.e. distributional) solutions of the Klein-Gordon equation. For us it is important to remember this proposition when we introduce the free quantum field in the next section.

Proposition 2.12. *Consider $(b,\Lambda)\in\mathcal{P}_+^\uparrow$ and the action of $\mathcal{P}_+^\uparrow$ on functions $f:\mathbb{R}^4\to\mathbb{C}$ given by $f\mapsto (b,\Lambda)f$ with $[(b,\Lambda)f](\xi) = f\big(\Lambda^{-1}(\xi-b)\big)$. The map $\mathscr{S}(H_m)\ni A\mapsto \mathrm{Re}\,\mathcal{F}\overline{A}$ intertwines this action with the representation $U(b,\Lambda)$ introduced in Proposition 2.3. In other words*

$$(b,\Lambda)\,\mathrm{Re}\,\mathcal{F}\overline{A} = \mathrm{Re}\,\mathcal{F}\overline{U(b,\Lambda)A}$$

holds for all $(b,\Lambda)\in\mathcal{P}_+^\uparrow$ and all $A\in\mathscr{S}(H_m)$.

Proof. We write $\Psi(\xi) = \mathrm{Re}(\mathcal{F}\overline{A})(\xi)$ and calculate

$$\begin{aligned}\Psi(\Lambda^{-1}(\xi-b)) &= \frac{1}{(2\pi)^{3/2}}\int_{H_m}\left(\overline{A(\lambda)}e^{i\eta(\Lambda^{-1}(\xi-b),\lambda)} + A(\lambda)e^{-i\eta(\Lambda^{-1}(\xi-b),\lambda)}\right)d\Omega_m(\lambda)\\ &= \frac{1}{(2\pi)^{3/2}}\int_{H_m}\left(\overline{A(\lambda)}e^{i\eta(\xi-b,\Lambda\lambda)} + A(\lambda)e^{-i\eta(\xi-b,\Lambda\lambda)}\right)d\Omega_m(\lambda).\end{aligned}$$

By substituting $\Lambda\lambda\to\tilde{\lambda}$, due to Lorentz invariance of the measure Ω_m we get

$$\begin{aligned}\Psi(\Lambda^{-1}(\xi-b)) &= \frac{1}{(2\pi)^{3/2}}\int_{H_m}\left(\overline{A(\Lambda^{-1}\tilde{\lambda})}e^{i\eta(\xi-b,\tilde{\lambda})} + A(\Lambda^{-1}\tilde{\lambda})e^{-i\eta(\xi-b,\tilde{\lambda})}\right)d\Omega_m(\tilde{\lambda})\\ &= \frac{1}{(2\pi)^{3/2}}\int_{H_m}\Big(e^{i\eta(x,\tilde{\lambda})}\underbrace{\overline{e^{i\eta(b,\tilde{\lambda})}A(\Lambda^{-1}\tilde{\lambda})}}_{=\overline{(U(b,\Lambda)A)(\tilde{\lambda})}} + \dots\Big)\,d\Omega_m(\tilde{\lambda})\\ &= \mathrm{Re}\left(\mathcal{F}\overline{U(b,\Lambda)A}\right)(\xi)\end{aligned}$$

which concludes the proof. □

Remark 2.13 (Canonical formalism)**.** We can regard the space $\mathcal{M} = \mathscr{S}(\mathbb{R},\mathbb{R})\times\mathscr{S}(\mathbb{R},\mathbb{R})$ of initial data as the classical phase space of the system. The classical Hamilton function

is given by

$$\mathcal{M} \ni (f,p) \mapsto \mathfrak{h}(f,p) = \frac{1}{2}\langle p,p\rangle + \frac{1}{2}\langle (m^2-\Delta)f, f\rangle \in \mathbb{R}$$

where $\langle\cdot,\cdot\rangle$ denotes the *real* standard scalar product on $\mathscr{S}(\mathbb{R}^3,\mathbb{R})$, i.e. $\langle f,g\rangle = \int_{\mathbb{R}^3} f(x)g(x)dx$. Using Gâteaux differentials we can easily define partial derivatives of $\mathfrak{h}$ as

$$\frac{\partial}{\partial f}\mathfrak{h}(f,p) \in \mathscr{S}(\mathbb{R}^3) \quad \text{with} \quad \Big\langle \frac{\partial}{\partial f}\mathfrak{h}(f,p), v\Big\rangle = \frac{d}{d\lambda}\mathfrak{h}(f+\lambda v,p)\big|_{\lambda=0}$$
$$\frac{\partial}{\partial p}\mathfrak{h}(f,p) \in \mathscr{S}(\mathbb{R}^3) \quad \text{with} \quad \Big\langle \frac{\partial}{\partial p}\mathfrak{h}(f,p), w\Big\rangle = \frac{d}{d\lambda}\mathfrak{h}(f,p+\lambda w)\big|_{\lambda=0}$$

with $v,w \in \mathscr{S}(\mathbb{R}^3)$. It is easy to see that these partial differentials exist in the given sense and have the values

$$\frac{\partial}{\partial f}\mathfrak{h}(f,p) = (\Delta^2 - m^2)f \quad \text{with} \quad \frac{\partial}{\partial p}\mathfrak{h}(f,p) = p.$$

Now consider a curve $\mathbb{R} \ni t \mapsto (f_t,p_t) \in \mathcal{M}$ which is differentiable in the sense that $(t,x) \mapsto f_t(x)$ and $(t,x) \mapsto p_t(x)$ are C^1-functions on $\mathbb{R}^4$. Then we can define the time derivative $(\dot{f}_t,\dot{p}_t) \in \mathcal{M}$ in terms of partial derivatives

$$\dot{f}_t(x) = \frac{\partial}{\partial t}f_t(x), \quad \dot{p}_t(x) = \frac{\partial}{\partial t}p_t(x),$$

such that the Hamilton equations become

$$\dot{f}_t = \frac{\partial}{\partial p}\mathfrak{h}(f_t,p_t) = p_t \qquad \dot{p}_t = -\frac{\partial}{\partial f}\mathfrak{h}(f_t,p_t) = (m^2-\Delta^2)f_t. \tag{2.14}$$

Differentiating a second time and inserting the second equation into the first shows that $(t,x) \mapsto f(t,x)$ has to satisfy the Klein-Gordon equation. Thus by Proposition 2.9 we can conclude that the system of equations (2.14) has a unique solution for all initial data $f_0 = f$, $p_0 = p$. We just take the unique solution $\psi \in \mathcal{K}$ of the Klein-Gordon equation with initial data f,p and define $f_t(x) = \psi(t,x)$ and $p_t(x) = \partial_t\psi(t,x)$. Hence Hamiltons equation with the Hamilton function $\mathfrak{h}$ represents a reinterpretation of the Cauchy problem for the Klein-Gordon equation as an infinite-dimensional Hamiltonian system, i.e. something like infinite-dimensional classical mechanics.

We are discussing this topic since we want to justify the interpretation of $\mathcal{M}$ as the classical phase space of our system, and f,p as the canonical variables. With substantially more effort we could also (in a mathematically rigorous way) introduce Poisson brackets and see that f,p satisfy some kind of "canonical" Poisson relations (maybe I will add this in a future version of these notes). This reinterpretation in a "canonical" (i.e. Hamiltonian) way is useful in the context of quantization. A possible strategy to quantize the field system we replace the classical fields $f(x)$ and $p(x)$ by operator-valued fields $\varphi(x),\pi(x)$ satisfying a version of "canonical commutation relations", and generate the Hamiltonian H of the quantum theory from $\mathfrak{h}$ by replacing f and p with φ and π. The spacetime field

$\Phi(t,x)$ then is generated by evolving $\varphi(x)$ with time evolution $\exp(itH)$.

While this procedure basically works (although with a substantial reinterpretation, which is nesscessary in order to make the math work rigorously) this is not the most adequate way to proceed. At that point the mass shell function $A_{f,p} \in \mathscr{S}(H_m) \subset \mathrm{L}^2(H_m, \Omega_m)$ comes into play. Since the map $(f,p) \mapsto A_{f,p}$ is invertible, we can regard the space $\mathscr{S}(H_m)$ as an alternative version of the phase space, which is parametrized in terms of complex variables $A, \overline{A}$ rather then f, p. Everything we have expressed in terms of f, p can be reexpressed with $A, \overline{A}$. E.g. the Hamilton function $\mathfrak{h}$ can be written as a function of $a = JA$ and $\bar{a}$ as follows.

$$\mathfrak{h}(a,\bar{a}) = \frac{1}{2}\int_{\mathbb{R}^3} \omega(k)\big(\bar{a}(k)a(k) + a(k)\bar{a}(k)\big)dk \tag{2.15}$$

It is left as an exercise to the reader to check this equation and to translate it into an integral over H_m involving $A, \overline{A}$ rather than $a, \bar{a}$.

The advantage of the variables $A, \overline{A}$ over f, p is the Poincaré covariance. The definition of the initial data f, p requires a split of spacetime into space and time, and this fixes an inertial frame up to spatial rotations and translations. Hence, there is not one canonical formalism, but there is (roughly speaking) a different one for each inertial frame. To understand why this is different for the mass shell functions $A, \overline{A}$, consider two inertial frames Σ_1, Σ_2 and a Poincaré transformation (b, Λ) transforming from Σ_1 into Σ_2. If a field is described in Σ_1 by a spacetime function ψ it is described in Σ_2 by $(b,\Lambda)\psi$. Hence if ψ is given by $A \in \mathscr{S}$ via $\psi = \sqrt{2}\,\mathrm{Re}\,\mathcal{F}\overline{A}$ the transformed field is given by $U(b,\Lambda)A$; cf. Proposition 2.12. Therefore the space $\mathscr{S}(H_m)$ does not only contain one but *all* canonical descriptions for all inertial frames and the unitaries $U(b,\Lambda)$ represent the transformation from ψ to $(b,\Lambda)\psi$. This intrinsic covariance is a great advantage if we are aiming at a quantum theory which has a similar covariance – like a Wightman quantum field theory.

Our strategy to quantize the Klein-Gordon equation is therefore to replace the functions $A, \overline{A}$ in all expressions developed so far by appropriately chosen operator-valued fields (satisfying some commutation relations which are motivated by the Poisson relations I have skipped). Applying this strategy in particular to the expression in (2.10) leads to an operator-valued spacetime field which (with an appropriate mathematical reinterpretation of the steps just outlined) will become our Wightman field.

2.3 The Free Quantum Field

First we need to make some general remarks on Fock spaces[12]. Let $\mathcal{H}$ be a separable Hilbert space. The belonging Fock space then is given by

$$\mathcal{F}(\mathcal{H}) = \mathbb{C} \oplus \Big(\bigoplus_{n=1}^{\infty} \mathcal{H}^{\otimes n}\Big).$$

[12] For more on the topic of Fock spaces of relativistic particles, we refer to [Bog+90, Chapter 7.3]

Now for $\sigma \in S_n$, where S_n is the permutation group on n, we define

$$V_\sigma : \mathcal{H}^{\otimes n} \to \mathcal{H}^{\otimes n} \qquad \sigma(\Psi_1 \otimes \ldots \otimes \Psi_n) = \Psi_{\sigma^{-1}(1)} \otimes \ldots \otimes \Psi_{\sigma^{-1}(n)}$$

and based on this, the symmetrization operator

$$S_n : \mathcal{H}^{\otimes n} \to \mathcal{H}_+^{\otimes n} \subset \mathcal{H}^{\otimes n} \qquad S_n = \frac{1}{n!} \sum_{\sigma \in S_n} V_\sigma.$$

Note, that S_n is a projection. Now the range $\mathcal{H}_+^{\otimes n}$ of S_n is called symmetric or Bose subspace. With this, we further define the symmetric Fock space as follows.

$$\mathcal{F}_+(\mathcal{H}) = \mathbb{C} \oplus \Big(\bigoplus_{n=1}^{\infty} \mathcal{H}_+^{\otimes n} \Big)$$

With this reminder, we can return to the free quantum field. We orient ourselves towards [RS75, Chapter X.7]. For some $f \in \mathcal{H}$ we define

$$b^-(f) : \mathcal{H}^{\otimes n} \to \mathcal{H}^{\otimes (n-1)} \qquad \Psi_1 \otimes \ldots \otimes \Psi_n \longmapsto \langle f, \Psi_1 \rangle \Psi_2 \otimes \ldots \otimes \Psi_n$$

A short calculation shows $\|b^-(f)\| = \|f\|$ so b^- is bounded and it can be linearly extended to all of $\mathcal{H}^{\otimes n}$ for all $n > 0$. For $n = 0$ we define $b^-(f) = 0$. It is easy to check that $b^+(f) = (b^-(f))^*$ on product vectors acts like

$$b^+(f) = (\Psi_1 \otimes \ldots \otimes \Psi_n) = f \otimes \Psi_1 \otimes \ldots \otimes \Psi_n.$$

With this, we want to define creation and annihilation operators as already known from the quantum harmonic oscillator.

For a densely defined self-adjoint operator $H : D(H) \to \mathcal{H}$ on $\mathcal{H}$ we explain

$$F_0 = \{\Psi \in \mathcal{F}_+(\mathcal{H}) \,|\, \exists_{n \in \mathbb{N}} \forall_{m > n} \Psi^{(m)} = 0\}$$
$$\tilde{D}_H = \{\Psi \in F_0 \,|\, \Psi^{(n)} \in D(H) \otimes \ldots \otimes D(H) \ \forall_n\}$$

where F_0 is the set of finite particle vectors. For $\Psi \in \tilde{D}_H$ we define

$$(d\Gamma(H)\Psi)^{(n)} = (H \otimes \mathbb{1} \otimes \ldots \otimes \mathbb{1} + \mathbb{1} \otimes H \otimes \mathbb{1} \otimes \ldots \otimes \mathbb{1} + \ldots)\Psi^{(n)}.$$

This is called second quantization. A similar construction can be done for unitaries U on $\mathcal{H}$ where

$$(\Gamma(U)\Psi)^{(n)} = (U \otimes \ldots \otimes U)\Psi^{(n)}$$

is said second quantization with

$$\Gamma(e^{itH}) = e^{it\, d\Gamma(H)}.$$

Now in order to define creation and annihilation operators, we need the number operator which is given by $N = d\Gamma(\mathbb{1})$. For $\Psi \in \mathcal{H}_+^{\otimes n}$ we obviously have $N\Psi = n\Psi$. Note that N is essentially self-adjoint on F_0. Sadly, N is a global observable whereas the lab setting obviously is local so on its own it is not as useful. This is why we define the annihilation operator on $\mathcal{F}_+(\mathcal{H})$ with domain F_0 to be

$$A(f) = \sqrt{N + \mathbb{1}}b^-(f).$$

For each $\Psi, \Phi \in F_0$ we have

$$\langle \Psi, A(f)\Phi\rangle = \langle \Psi, \sqrt{N+\mathbb{1}}b^-(f)\Phi\rangle = \langle \sqrt{N+\mathbb{1}}\Psi, b^-(f)\Phi\rangle = \langle S(b^-(f))^*\sqrt{N+\mathbb{1}}\Psi, \Phi\rangle$$

which implies that creation operator is given by

$$A^*(f) = (A(f))^* = S(b^-(f))^*\sqrt{N+\mathbb{1}} = \sqrt{N}Sb^+(f).$$

It is worth noting that both $A(f)$ and $A^*(f)|_{F_0}$ are closable. We denote their closures by $A(f)$ and $A^*(f)$.

Example 2.14. Consider a topological space with Borel measure (M, μ) and the associated Hilbert space $\mathcal{H} = \mathrm{L}^2(M, \mu)$. By [RS80, Chapter II.4], we then have

$$\mathcal{H}^{\otimes n} = \mathrm{L}^2(\underbrace{M \times \ldots \times M}_{n \text{ arguments}}, \mu \otimes \ldots \otimes \mu) \qquad \mathcal{H}_+^{\otimes n} = \mathrm{L}^2_S(M \times \ldots \times M, \mu \otimes \ldots \otimes \mu)$$

where L^2_S is the set of symmetric functions[13]. The operators A and A^* are given by

$$(A(f)\Psi)^{(n)}(m_1, \ldots, m_n) = \sqrt{n+1}\int_M \overline{f(m)}\Psi^{(n+1)}(m, m_1, \ldots, m_n)\, d\mu(m)$$

$$(A^*(f)\Psi)^{(n)}(m_1, \ldots, m_n) = \frac{1}{\sqrt{n}}\sum_{j=1}^{n} f(m_j) \cdot \Psi^{(n-1)}(m_1, \ldots, \underset{\text{skip}}{\hat{m}_j}, \ldots, m_n).$$

Definition 2.15. *The Segal quantization over $\mathcal{H}$ on F_0 is defined via*

$$\mathcal{H} \ni f \longmapsto \Phi_S(f) = \frac{1}{\sqrt{2}}(A(f) + A^*(f)) \in \mathcal{F}_+(\mathcal{H}).$$

Note that the Segal quantization is $\mathbb{R}$-, but not $\mathbb{C}$-linear since $f \mapsto b^-(f)$ is an antilinear map.

Theorem 2.16 ([RS75], Theorem X.41)**.** *The following statements hold.*

(a) (Self-adjointness). For all $f \in \mathcal{H}$, $\Phi_S(f)$ is essentially self-adjoint on F_0.

[13] This means that L^2_S is the set of functions in L^2 which are invariant under permutations of the coordinates.

(b) (Cyclicity of the vacuum). For the vacuum $\Omega = 1 \oplus 0 \oplus 0 \oplus \dots$

$$\operatorname{span}\{\Phi_S(f_1)\cdots\Phi_S(f_n)\Omega \,|\, f_1,\dots,f_n \in \mathcal{H}, n \in \mathbb{N}\} \subset \mathcal{F}_+(\mathcal{H})$$

is dense.

(c) (Commutation relations). For each $\Psi \in F_0$, $f, g \in \mathcal{H}$ *we have*

$$[\Phi_S(f), \Phi_S(g)]\Psi = i \operatorname{Im}\langle f, g\rangle_{\mathcal{H}}\Psi.$$

Further, for the unitary operator $w(f) = \exp(i\Phi_s(f))$ *(Weyl operator) we have*

$$W(f+g) = e^{-i\operatorname{Im}\langle f,g\rangle}W(f)W(g).$$

(d) (Continuity). If $f_n \to f$ *in* $\mathcal{H}$, *then*

$$\begin{aligned} W(f_n)\Psi &\to W(f)\Psi \quad \text{for all } \Psi \in \mathcal{F}_+(\mathcal{H}) \\ \Phi_S(f_n)\Psi &\to \Phi_s(f)\Psi \quad \text{for all } \Psi \in F_0 \end{aligned}$$

(e) (Covariance condition). For all unitary operators U *on* $\mathcal{H}$,

$$\Gamma(U) : D(\overline{\Phi_S(f)}) \to D(\overline{\Phi_S(Uf)})$$

and for $\Psi \in D(\overline{\Phi_S(Uf)})$ *we have*

$$\Gamma(U)(\overline{\Phi_S(f)})\Gamma(U)^*\Psi = \overline{\Phi_S(Uf)}\Psi$$

for all $f \in \mathcal{H}$.

We can now use the Segal quantization to define the free Hermitian scalar field of mass m. To this end note that we can write $\Phi_S = \sqrt{2}\operatorname{Re}A^*$ which is (almost) the same expression we have already used in Proposition 2.11 to rewrite solutions to the Klein-Gordon equation in a distributional sense. The only difference is the quantity A which is now an operator rather than a function (and complex conjugation becomes taking adjoint). Hence with the map E from Equation (2.13), which was given by

$$E : \mathscr{S}(\mathbb{R}^4) \to \mathrm{L}^2(H_m) \qquad f \longmapsto \sqrt{2\pi}\,\tilde{f}|_{H_m}$$

using the 4-dimensional Fourier transform

$$\tilde{f}(k) = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^4} f(x)e^{i\eta(x,k)}\,dx,$$

the free Hermitian scalar field of mass m now is defined to be

$$\Phi_m(f) = \Phi_S(Ef) = \sqrt{2}\operatorname{Re}A^*(Ef)$$

on $f \in \mathscr{S}(\mathbb{R}^4, \mathbb{R})$ and

$$\Phi_m(f) = \Phi_S(\mathrm{Re}(f)) + i\Phi_S(\mathrm{Im}(f))$$

for arbitrary $f \in \mathscr{S}(\mathbb{R}^4)$. Applying the reasoning from Proposition 2.11 we see that *formally* this $\Phi_m(f)$ is an operator-valued solution of the Klein-Gordon equation. Exact reasoning leads to the following theorem.

Theorem 2.17 ([RS75], Theorem X.42)**.** *The 5-tuple*

$$\Big(\mathcal{H}_m, F_0, \Phi_m, \Omega, \Gamma\big(U(\cdot,\cdot)\big)\Big)$$

is a Wightman quantum field where $\mathcal{H}_m = \mathcal{F}_+(\mathrm{L}^2(H_m))$ is the Hilbert space of the free field and

$$\big(U(b,\Lambda)\Psi\big)(p) = e^{i\eta(\Lambda,b)}\Psi(\Lambda^{-1}p)$$

is the unitary representation of the restricted Poincaré group on $\mathrm{L}^2(H_m, \Omega_m)$. Further

$$\Phi_m\big((\Box + m^2)f\big) = 0$$

holds for each $f \in \mathscr{S}(\mathbb{R}^4)$.

Proof. Most of the statement can be derived easily from earlier results, like the discussion of the Klein-Gordon equation in Sec. 2.2 or properties of the Segal quantization in Theorem 2.16. Only a few properties (like self-adjointness) require more work. In those cases we refer the reader to the corresponding proof in [RS75]. □

Here, Φ_m is also called the free field.

Remark 2.18 (Free Hamiltonian)**.** Following the reasoning from Remark 1.25 we can define the free Hamiltonian of the theory (in the inertial system in which we are at rest) as the generator of the time translations, i.e. as the self-adjoint operator H_0 satisfying

$$\Gamma\big(U(te_0, \mathbb{1})\big) = \exp(itH_0), \quad \forall t \in \mathbb{R}.$$

Using the properties of second quantization $\Gamma(U)$ this can be rewritten as

$$H_0 = d\Gamma(P_0) \quad \text{with} \quad (P\psi)(\omega(k), k) = \omega(k)\psi(k, \omega(k)),$$

where $\psi \in \mathrm{L}^2(H_m, \omega_m)$ has to be chosen such that $P\psi$ is square-integrable again. This gives the domain of P_0. We come back to H_0 in Remark 2.24, where we derive a different expression for it which is more familiar from the physics literature.

In Sec. 2.2 we have used the freedom to rewrite functions on the mass shell as functions on $\mathbb{R}^3$ by using the unitary operator $J : \mathrm{L}^2(H_m, \Omega_m) \to \mathrm{L}^2(\mathbb{R}^3)$ from Proposition 2.5. We

can do the same with the field operators by applying the second quantization of J to $\Phi_m(f)$. For real-valued f this leads to

$$\tilde{\Phi}_m(f) = \Gamma(J)\Phi_m(f)\Gamma(J)^* = \frac{1}{\sqrt{2}}\left(a(Ef) + a^*(Ef)\right)$$

with the creation and annihilation operators a^*, a on $\mathcal{F}_+\big(\mathrm{L}^2(\mathbb{R}^3)\big)$. We have used lower case letters here, in order to make it easier to distinguish them from the corresponding operators on $\mathcal{F}_+\big(\mathrm{L}^2(H_m, \Omega)\big)$. The relations to the latter are

$$\Gamma(J)A(f)\Gamma(J)^* = a\Big(\frac{f \circ j}{\sqrt{\omega}}\Big) \qquad \Gamma(J)(A(f))^*\Gamma(J)^* = a^*\Big(\frac{f \circ j}{\sqrt{\omega}}\Big).$$

Note that these are operators on $\mathcal{F}_+\big(\mathrm{L}^2(\mathbb{R}^3)\big)$. As before the finite particle vectors $F_0 \subset \mathcal{F}_+\big(\mathrm{L}^2(\mathbb{R}^3)\big)$ form the domain of these operators. In slight abuse of notation we here have kept the same symbol (F_0) we have already used for the finite particle vectors in $\mathcal{F}_+\big(\mathrm{L}^2(H_m, \Omega_m)\big)$, although strictly speaking both sets are different.

We now turn to the question whether we can evaluate $\tilde{\Phi}_m$ at spacetime events rather than test functions. The answer, basically, is yes, but the result of such an evaluation is not an operator. To explain the details we need some preparations. (Also note that we only treat the case $\tilde{\Phi}_m$. The mass shell based field Φ_m can be treated similarly. To work out the corresponding details is left as an exercise to the reader).

On the domain

$$D_{\mathscr{S}} = \{\Psi \in F_0 \,|\, \Psi^{(n)} \in \mathscr{S}((\mathbb{R}^3)^n) \text{ for all } n\} \tag{2.16}$$

we can define the annihilation operator a_k at momentum k via

$$(a(k)\Psi)^{(n)}(k_1, \ldots, k_n) = \sqrt{n+1}\,\Psi^{(n+1)}(k, k_1, \ldots, k_n), \tag{2.17}$$

which is a well-defined operator and related to $a(f)$ with test function $f \in \mathrm{L}^2(\mathbb{R}^3)$ by the following Proposition.

Proposition 2.19. *For all $f \in \mathrm{L}^2(\mathbb{R}^3)$ the annihilation operator $a(f)$ can be written as*

$$a(f) = \int_{\mathbb{R}^3} \overline{f(k)} a_k \, dk$$

with a_k from Eq. (2.17).

Proof. This is straightforward and again left as an exercise. □

Basically, we would like to do something similar, but if we try to calculate the adjoint a_k^* of a_k *formally* we get

$$(a^*(k)\Psi)^{(n)}(k_1, \ldots, k_n) = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \delta(k - k_j)\Psi^{(n-1)}(k_1, \ldots, \hat{k}_j \ldots, k_n) \tag{2.18}$$

with the obvious problem that δ occurs, so $a^*(\cdot)$ is not a densely defined operator. We can rectify this by using quadratic forms.

Remark 2.20 (Quadratic forms. [RS80], Sec. VIII.6)**.** A quadratic form q on a Hilbert space $\mathcal{H}$ is a map $q : Q(q) \times Q(q) \to \mathbb{C}$, where $Q(q) \subset \mathcal{H}$ is a dense, linear subspace (the *form domain*), and q is linear in the second and conjugate linear in the first argument. If X is an operator with dense domain $D(X)$ we can immediately define a quadratic form $X[\phi, \psi] = \langle \phi, X\psi \rangle$ with domain $D(X)$. The converse is not true. There are quadratic forms which do not belong to an operator (we will see an example very soon). Hence, quadratic forms are more singular objects than operators. Nevertheless the notation $\langle \psi, q\phi \rangle$ is frequently used for $q[\psi, \phi]$ even if q is not an operator. In the following we will use the phrase "X and q coincide in the sense of quadratic forms" for an operator X and a quadratic form q if $q[\psi, \phi] = \langle \psi, X\phi \rangle$ holds for all ψ, ϕ in an appropriate domain.

Now assume a_k^* would exist as an operator. Then we could assign it to the quadratic form

$$a_k^*[\psi, \phi] = \langle \psi, a_k^* \phi \rangle = \langle a_k \psi, \phi \rangle \quad \psi, \phi \in D_{\mathscr{S}}$$

Hence, although a_k^* is not an operator the quantity on the right-hand side of this equation only uses a_k and is therefore well-defined. We use this expression as the definition of a_k^* *as a quadratic form*. Now in analogy to Proposition 2.19 we get the following.

Proposition 2.21. *For all $f \in \mathrm{L}^2(\mathbb{R}^3)$ we have*

$$a^*(f) = \int_{\mathbb{R}^3} f(k) a_k^* \, dk$$

in the sense of quadratic forms.

Proof. Again, this is straightforward. □

Now, we can apply this to the field $\tilde{\Phi}_m$ and get the following result.

Theorem 2.22. *The quantity*

$$\Phi_m(t, x) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} \left(e^{i(\omega(k)t - kx)} a_k^* + e^{-i(\omega(k)t - kx)} a_k \right) \frac{d^3k}{\sqrt{2\omega(k)}} \tag{2.19}$$

is a well-defined quadratic form on $\mathcal{F}_+\big(\mathrm{L}^2(\mathbb{R}^2)\big)$ with domain $D_{\mathscr{S}}$. For a test function $f \in \mathscr{S}(\mathbb{R}^4)$ it is related to the field $\tilde{\Phi}_m(f)$ by

$$\tilde{\Phi}_m(f) = \int_{\mathbb{R}^4} f(x, t) \Phi_m(x, t) dx \, dt.$$

Proof. The fact that $\Phi_m(t, x)$ is a well-defined quadratic form is obvious from the definitions. The relation to $\Phi_m(f)$ can be cheked with the arguments already used in the proof of Proposition 2.11. [In a future version of these notes I might expand this proof.] □

Remark 2.23 (Wick ordering)**.** For this remark, cf. [Haa96, Chapter I.5.2]. The idea to use quadratic forms in order to evaluate quantities at spacetime events can be extended to powers of the field like $\Phi^n(t,x)$ or more generally expressions involving polynomials in the a_k, a_k^*. If we look in particular at $(a_k^*)^n(a_k)^m$ we can easily use the same idea as above and just "move all a_k^* to the left under the scalar product":

$$\langle \phi, (a_k^*)^n(a_k)^m\psi\rangle = \langle (a_k)^n\phi, (a_k)^m\psi\rangle =: (a_k^*)^n(a_k)^m[\phi,\psi]. \tag{2.20}$$

If we change the ordering of the a_k, a_k^*, however this procedure fails. In the product $a_k a_k^*$ we can not move a_k^* "to the left" since a_k and a_k^* are not commuting. Hence we have to move a_k first which would produce an (undefined) operator a_k^* in the left argument of the scalar product. To understand the problem we are facing let us see what happens if we calculate the vaccum expectation value $\langle\Omega, a_k a_k^*\Omega\rangle$ of $a_k a_k^*$ with the formal expression from Eq. (2.18).

$$\langle \phi, a_k a_k^*\psi\rangle = \langle a_k^*\Omega, a_k^*\Omega\rangle = \int_{\mathbb{R}^3} \delta^2(k-p)dp.$$

The integral on the right-hand side involves the square of the delta function which can not be defined within distribution theory. Hence, the given vacuum expectation value is just infinite. Usually, physical quantities like energy should have vacuum expectation value zero (naively speaking, the vacuum should just mean no physical particles). In that sense the infinities we see are just artifacts arising from wrong operator ordering. To get the real physical quantities we just have to subtract these infinities, and this can be done by choosing the correct operator ordering. The only ordering where no problems in terms of inifinities arise is the one in Eq. (2.20). Hence the simple rule is: Whenever we encounter a monomial in a_k, a_k^* is: "move all creation operators to the left". This is known as Wick or normal ordering and usually indicated by colons written to the left and the right of an expression. E.g. to calculate $:\Phi(t,x)^n:$ we formally expand $\Phi(t,x)^n$ into a polynomial of a_k, a_k^* and in any monomial we get that way, we move all a_k^* to the left.

Remark 2.24 (Operator ordering and quantization)**.** Operator ordering problems as the ones just described are not uncommon in quantum theory and already happen within the quantization of one non-relativistic particle. The (too) simple rule which tries to map classical observables (functions on phase space) to quantum observables (operators on a Hilbert space) is to replace the canonical phase space coordinates q_j, p_k by position and momentum operators Q_j, P_k and to replace each occurence of q_j, p_k in a phase space function F by these operators. But even if F is just a polynomial in q_j, p_k this procedure is ambiguous since the q_k, p_k mutually commute such that $q_j p_j = p_j q_j$ while $Q_j P_j$ and $P_j Q_j$ are different operators. Hence, we need additional (physical) arguments in order to make the operator ordering unambiguous. In the given example we might want to use $\frac{1}{2}(Q_j P_j + P_j Q_j)$ since this combination is at least hermitian.

Based on these considerations we can describe the basic quantization rule for scalar fields as follows:

1. Rewrite a classical field observable (i.e. a function $F : \mathcal{M} \to \mathbb{R}$ on classical phase

space $\mathcal{M}$; cf. the disicussion in Remark 2.13) as a polynomial in the complex valued phase space variables $a(k), \bar{a}(k)$.

2. Replace these functions by the creation and annihilation operators (or more precisely quadratic forms) $\tilde{a}_k, \tilde{a}_k^*$. (Note that we are adding a tilde to the operators here in order to make the distinction from the functions $a, \bar{a}$ easier. We will drop this tilde after this remark is finished).
3. Apply Wick ordering, in order to get rid of operator ordering ambiguities and the corresponding infinities. The choice of $\tilde{a}_k$, $\tilde{a}_k^*$ as the proper replacement of $a(k), \bar{a}(k)$ is motivated by a comparison of the commutation relations between $\tilde{a}_k$, $\tilde{a}_k^*$ on the one hand and the Poisson relations between $a(k)$ and $\bar{a}(k)$ on the other.

This receipt provides a clear rule to turn any polynomial in $a(k)$, $\bar{a}(k)$ into a quadratic form on the Hilbert space $\mathcal{F}_+\big(\mathrm{L}^2(\mathbb{R}^3)\big)$. If we apply it to the classical solution of the Klein-Gordon Equation from (2.8) we get the quantum field $\Phi(t,x)$ as written in Eq. (2.19). Another possible application is the classical Hamilton function $\mathfrak{h}(a,\bar{a})$ from Eq. (2.15).

$$:\mathfrak{h}(\tilde{a}_k, \tilde{a}_k^*): \;= \int_{\mathbb{R}^3} \omega(p) : \underbrace{\frac{1}{2}(\tilde{a}_k^* \tilde{a}_k + \tilde{a}_k \tilde{a}_k^*)}_{\text{harmonic oscillator}} : dp = \int_{\mathbb{R}^3} \omega(k) \tilde{a}_k^* \tilde{a}_k \, dp = H_0$$

Please check yourself that (as a quadratic form) this really coincides with the free Hamiltonian H_0 from Remark 2.18. Also note in this context that the number operator N can be given by a similar expression as

$$N = \int_{\mathbb{R}^3} \tilde{a}_p^* \tilde{a}_p \, dp.$$

Finally note that exactly the same discussion can be given in terms of the mass shell functions $A(\lambda), \overline{A}(\lambda)$ and the corresponding quadratic forms $\tilde{A}_\lambda$, $\tilde{A}_\lambda$ on $\mathcal{F}_+\big(\mathrm{L}^2(H_m, \Omega_m)\big)$. Working this out is left as an exercise.

For the last point in this section we have to talk about are the Wightman distributions of $\Phi_m(t,x)$. They are calculated in the next theorem.

Theorem 2.25. *For $m, n \in \mathbb{N}_0$ with $m > 1$ the Wightman distributions of the free field Φ_m from Theorem 2.17 are given by $W^{(2n+1)} = 0$ and*

$$W^{(2m)}(f_1 \otimes \ldots \otimes f_{2m}) = \sum_{\sigma \in pair} W^{(2)}(f_{\sigma(1)} \otimes f_{\sigma(2)}) \ldots W^{(2)}(f_{\sigma(2n-1)} \otimes f_{\sigma(2m)}) \quad (2.21)$$

where $pair \subset S_{2n}$ is the set of permutations which satisfy $\sigma(1) < \sigma(3) < \ldots < \sigma_{(2n-1)}$ and $\sigma_{(2k+1)} < \sigma_{(2k+2)}$ for $k > 0$. The two-point function $W^{(2)}$ is given by

$$W^{(2)}(f \otimes g) = \frac{1}{2(2\pi)^3} \int_{H_m} \int_{\mathbb{R}^4} \int_{\mathbb{R}^4} e^{i\eta(y-x,p)} f(x) g(y) \, dx \, dy \, d\Omega_m(p),$$

which can formally be rewritten as

$$W^{(2)}(f \otimes g) = \frac{1}{i} \int_{\mathbb{R}^4} \int_{\mathbb{R}^4} \Delta_+(y-x, m^2) f(x) g(y)\, dx\, dy$$

where

$$\Delta_+(x, m^2) = \frac{i}{2(2\pi)^3} \int e^{i\eta(x,p)} \frac{d^3p}{\omega(p)}.$$

Proof. The combinatorial structure of the $W^{(n)}$ given here is a consequence of the structure of the Segal quantization and therefore not special to the free field. A corresponding proof can be given using Wick's theorem. Typically, this is left as an exercise. In lack of a good reference I am following this tradition. Please check validity of (2.21) yourself. To calculate the two-point function we use $Ef = \sqrt{2\pi}\tilde{f}$ and further get

$$\begin{aligned}
W^{(2)}(f \otimes g) &= \langle \Omega_m, \Phi_m(f)\Phi_m(g)\Omega_m \rangle = \langle \Phi_m(\overline{f})\Omega_m, \Phi_m(g)\Omega_m \rangle \\
&= \frac{2\pi}{2} \langle a^*(\overline{\tilde{f}}), a^*(\hat{g}) \rangle = \frac{2\pi}{2} \langle \overline{\tilde{f}}, g \rangle_{\mathrm{L}^2(H_m)} \\
&= \frac{2\pi}{2} \int_{H_m} \frac{1}{(2\pi)^2} \int_{\mathbb{R}^4} e^{-i\eta(x,p)} f(x)\, dx \, \frac{1}{(2\pi)^2} \int_{\mathbb{R}^4} e^{i\eta(y,p)} g(y)\, dy\, d\Omega_m(p) \\
&= \frac{1}{2(2\pi)^3} \int_{H_m} \int_{\mathbb{R}^4} \int_{\mathbb{R}^4} e^{i\eta(y-x,p)} f(x) g(y)\, dx\, dy\, d\Omega_m(p)
\end{aligned}$$

Written formally, this yields

$$W^{(2)}(f \otimes g) = \frac{1}{i} \int_{\mathbb{R}^4} \int_{\mathbb{R}^4} \Delta_+(y-x, m^2) f(x) g(y)\, dx\, dy$$

where

$$\Delta_+(x, m^2) = \frac{i}{2(2\pi)^3} \int e^{i\eta(x,p)} \frac{d^3p}{\omega(p)},$$

as claimed. □

Fields with this structure of the n-point functions are called generalized free fields or quasi-free.

Theorem 2.26 (Källen-Lehmann-representation)**.** *Let $W^{(2)}$ be the two-point function of a field theory satisfying the Wightman axioms and the additional condition that*

$$\langle \psi_0, \varphi(f)\psi_0 \rangle = 0$$

for all $f \in \mathscr{S}(\mathbb{R}^4)$. Then there exists a polynomially bounded positive measure on $[0, \infty)$

so that for all $f \in \mathscr{S}(\mathbb{R}^4)$,

$$W^{(2)}(f) = \int_0^\infty \left(\int_{H_m} \tilde{f}\, d\Omega_m \right) d\rho(m).$$

Symbolically,

$$W^{(2)}(x) = \int_0^\infty \frac{1}{i} \Delta_+(x, m^2)\, d\rho(m).$$

Proof. [RS75, Theorem IX.34] □

2.4 Time-Zero Fields

Before we can define the time-zero field, we need the following short result.

Proposition 2.27. *For* $f \in \mathscr{S}(\mathbb{R}^3)$ *we define the dual space element* $\delta f \in \mathscr{S}'(\mathbb{R}^4)$ *via*

$$(\delta f)(g) := \int_{\mathbb{R}^4} \delta(t) g(t,x) f(x)\, dt\, dx = \int_{\mathbb{R}^3} g(0,x) f(x)\, dx.$$

Then the map E *from (2.13) can be extended to distributions of the form* δf.

Proof. Using the definition of δf we get

$$\begin{aligned}(E(\delta f))(\omega(k), k) &= \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^4} \delta(t) f(x) e^{i\omega(k)t} e^{-ik\cdot x}\, dt\, dx \\ &= \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} f(x) e^{-ik\cdot x}\, dx = \hat{f}(k),\end{aligned}$$

so $E(\delta f)(j(k)) = \hat{f}(k)$ with j from (2.2). □

Definition 2.28. *We define the time-zero fields to be*

$$\varphi_m(f) = \Phi_S(E(\delta f)) \qquad \pi_m(f) = \Phi_S(i\omega E(\delta f))$$

for $f \in \mathscr{S}(\mathbb{R}^3, \mathbb{R})$ *and its* $\mathbb{C}$*-linear extension onto all of* $\mathscr{S}(\mathbb{R}^3)$.

As in the spacetime field Φ_m we can use the unitary $\Gamma(j)$ to transform φ_m and π_m into operators on $\mathcal{F}_+\big(\mathrm{L}^2(\mathbb{R}^3)\big)$. We define

$$\tilde{\varphi}_m(f) = \Gamma(J)\varphi_m(f)\Gamma(J)^* \quad \text{and} \quad \tilde{\pi}_m(f) = \Gamma(J)\pi_m(f)\Gamma(J)^*.$$

In analogy to Theorem 2.22 we can express $\tilde{\varphi}_m(f)$ and $\tilde{\pi}_m(f)$ as "smeared out versions" of appropriately chosen quadratic forms.

Proposition 2.29. *The time-zero fields $\tilde{\varphi}_m$, $\tilde{\pi}_m$ can be written in terms of a_k and a_k^* as quadratic forms via*

$$\tilde{\varphi}_m(f) = \int f(x)\varphi_m(x)\,dx$$

where

$$\varphi_m(x) = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}\left(e^{-ik\cdot x}a_k^* + e^{ik\cdot x}a_k\right)\frac{dk}{\sqrt{2\omega(k)}}$$

and

$$\tilde{\pi}_m(g) = \int_{\mathbb{R}^3} g(x)\pi_m(x)\,dx$$

with

$$\pi_m(x) = \frac{i}{(2\pi)^{3/2}}\int_{\mathbb{R}^3}\omega(k)\left(e^{-ik\cdot x}a_k^* - e^{ik\cdot x}a_k\right)\sqrt{\frac{\omega(k)}{2}}\,d^3k.$$

Proof. In the case of $\tilde{\varphi}_m$ this follows immediately from Theorem 2.22 by setting $t = 0$. In the case of $\tilde{\pi}_m$ we use (2.19) and take a time derivative to get

$$\partial_t\Phi_m(t,x) = \frac{i}{(2\pi)^{3/2}}\int_{\mathbb{R}^3}\left(\omega(k)e^{i(\omega(k)t-kx)}a_k^* - \omega(k)e^{-i(\omega(k)t-kx)}a_k\right)\frac{d^3k}{\sqrt{2\omega(k)}}$$

and thus

$$(\partial_t\tilde{\Phi}_m)(f) = \tilde{\Phi}_m(i\omega f) = \Gamma(J)\Phi_S(\omega E(if))\Gamma(J)^+.$$

Hence, setting $t = 0$ again leads to the statement about $\tilde{\pi}_m$. □

Remark 2.30 (Time evolution)**.** We can look at the fields $\varphi_m(x)$ and $\pi_m(x)$ as local observables which we can evolve in time (i.e. we are in the Heisenberg picture). With the free Hamiltonian H_0 from Remark 2.18 we can define

$$e^{-itH_0}\varphi_m(f)e^{itH_0} = \varphi_{m,t}(f) \qquad e^{-itH_0}\pi_m(g)e^{itH_0} = \pi_{m,t}(g).$$

This again yields

$$\begin{aligned} e^{-itH_0}\varphi_m(x)e^{itH_0} &= \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}\Big(e^{-ik\cdot x}\underbrace{e^{-itH_0}a_k^*e^{itH_0}}_{=e^{it\omega(k)}a_k^*} + e^{ik\cdot x}e^{-itH_0}a_k e^{itH_0}\Big)\frac{dk}{\sqrt{2\omega(k)}} \\ &= \Phi_m(t,x). \end{aligned}$$

In other words we can reconstruct the spacetime field $\Phi_m(x,t)$ from the time-zero field and time evolution, which in turn is part of the representation $\Gamma\big(U(b,\Lambda)\big)$ of the Poincaré

group. This observation leads to the question whether we can turn this into a strategy to costruct new models from a set of time-zero fields and a representation of $\mathcal{P}_+^\uparrow$, or at least a Hamiltonian. The latter depends on physical input like the type of interactions we want to describe. For the fields, however, we'd like to have a universal construction, which is completely independent from any dynamical input. We might even hope that the time-zero fields developed for the free field are already sufficient. To explain up to which degree this idea works or does not work is the main task of the rest of this section.

Remark 2.31 (Canonical commutation relations)**.** To understand the last remark a bit better, let us have a short look at the quantum mechanical description of N non-relativistic particles. The Hilbert space of this system is $\mathrm{L}^2(\mathbb{R}^n)$ with $n = 3N$ and the most important observables are position and momentum described by self adjoint operators Q_j, P_k which are defined for $\psi \in \mathscr{S}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$ by

$$\big(Q_j\psi\big)(x) = x_j\psi(x), \quad \big(P_j\psi\big)(x) = \frac{1}{i}\frac{\partial\psi}{\partial x_j}(x). \tag{2.22}$$

The Q_j and P_k should be reagarded as a the finite degree of freedom analog of the time-zero fields $\varphi_m(x)$, $\pi_m(x)$. The most important structural property of the Q_j, P_k are the canonical commutation relations, which (in their most elementary form) are given as

$$[Q_jQ_k]\psi = [P_j, P_k]\psi = 0, \qquad [Q_j, P_k]\psi = i\delta_{jk}\psi \tag{2.23}$$

for any $\psi \in \mathscr{S}(\mathbb{R}^n)$. These relations are the main reason why the Q_j and P_k are chosen in the given way: We are following Dirac's quantization rule which says: "Quantization means to replace Poisson brackets by operator commutators". The Q_j and P_k should be the quantum analogs of position and momentum coordinates q_j, p_k of the classical phase space $\mathbb{R}^n \times \mathbb{R}^n$ and the relations in (2.23) exactly resemble the Poisson relations between q_j and p_k. Large parts of the physics literature even tell the legend that up to unitary equivalence the Q_j, P_k from Eq. (2.22) are the only possible choice for operators satisfying the commutation relations in (2.23). If we assume for the moment that this is true (we will come to that in a minute), and if we accept that the CCRs in (2.23) are mandatory requirements, we can conclude that the operators in (2.22) are the only possible choice – completely independent from the dynamics our model should obey. The latter is completely contained in the Hamiltonian of the system.

What can we learn from this for field theory? First of all note that the expressions for $\varphi_m(x)$ and $\pi_m(x)$ in Proposition 2.29 resemble Eqs. (2.9) where we have expressed the intial data $(f, p) \in \mathcal{M}$ for the classical solution in terms of the complex variables $a, \bar{a}$. According to the discussion in Remark 2.24 the fields $\varphi_m(x)$ and $\pi_m(x)$ are the quantizations of the canonical phase space variables $f(x)$ and $p(x)$. Since the latter are the field thoeretic replacement of phase space coordinates q_j, p_k (cf. our discussion in Remark 2.13) this observation justifies the claim that $\varphi_m(x)$ and $\pi_m(x)$ are the field theoretic analogs of the operators Q_j, P_k. Hence let us have a look at the commutation

relations they satisfy:

$$[\tilde{\varphi}_{m,t}(f), \tilde{\varphi}_{m,t}(g)]\psi = [\tilde{\pi}_{m,t}(f), \tilde{\pi}_{m,t}(g)]\psi = 0 \qquad [\tilde{\varphi}_{m,t}(f), \tilde{\pi}_{m,t}(g)] = i\langle f, g\rangle, \tag{2.24}$$

where on the right-hand side of the second equation we have used the scalar product in $\mathrm{L}^2(\mathbb{R}^3)$, and $\psi \in F_0$ is chosen arbitrarily. Rewriting this in a formal way we get the more often used form

$$[\varphi_{m,t}(x), \varphi_{m,t}(y)] = [\pi_{m,t}(x), \pi_{m,t}(y)] = 0 \qquad [\varphi_{m,t}(x), \pi_{m,t}(y)] = i\delta(x - y). \tag{2.25}$$

The latter version can be regarded as the natural "continuous variable version" of the relations in (2.23). Therefore we proceed in analogy to the finite degree of freedom case as follows. We declare the CCRs in Eq. (2.25), or maybe better the mathematically more rigorous version in (2.24), as the fundamental quantization condition each choice of time-zero fields (for scalar field theories) should satisfy. If there would be again a uniqueness result, the time-zero fields constructed for the free field would be the only choice and we could proceed along the lines outlined in Remark 2.30. The big amounts of subjunctives in the last sentence already indicate that there is a problem with this procedure to explain why we have to clarify first, in which sense the Q_j and P_k are really unique. This step is prepared by the following definition.

Definition 2.32. *Consider a locally convex, real vector space V which also carries a real scalar product $\langle \cdot, \cdot \rangle$, and strongly continuous maps $f \mapsto U(f)$, $g \mapsto V(g)$ from V into the unitary group $\mathcal{U}(\mathcal{H})$ of a separable Hilbert space $\mathcal{H}$. If for all $f_1, f_2 \in V$ the operators U, V satisfy the Weyl relations*

$$\begin{aligned} V(f_1 + f_2) &= V(f_1)V(f_2) \\ U(f_1 + f_2) &= U(f_1)U(f_2) \\ V(f)U(g) &= U(g)V(f)\exp(\langle f, g\rangle), \end{aligned}$$

then $\{U, V\}$ is called a representation of the Weyl relations over V.

Example 2.33 (Schrödinger representation)**.** The central reference for all statements concerning the Schrödinger representation is the book of Folland [Fol16]. Consider $V = \mathbb{R}^n$ and the operators Q_j, P_k from Eq. (2.22). Real linear combinations of the Q_j and of the P_k are self-adjoint operators (on appropriate domains containing $\mathscr{S}(\mathbb{R}^n)$ as a subspace). Hence with $f, g \in \mathbb{R}^n$ we can define

$$U(f) = \exp\left(i\sum_{j=1}^{n} f_j Q_j\right), \quad \text{and} \quad V(g) = \exp\left(i\sum_{k=1}^{n} g_k P_k\right). \tag{2.26}$$

By Stone's theorem [RS80, Sec. VIII.4] these maps are strongly continuous. It is also

easy to check that

$$\big(U(f)\psi\big)(x) = \exp\left(\sum_{j=1}^{n} x_j f_j\right)\psi(x), \quad \text{and} \quad \big(V(g)\psi\big)(x) = \psi(x+g)$$

hold. From the last equation it is easy to see that the $U(f)$, $V(g)$ satisfy the Weyl relations. Hence the maps $\mathbb{R}^n \ni f \mapsto U(f)$ and $\mathbb{R}^n \ni g \mapsto V(g)$ form a representation of the Weyl relation, which is called the *Schrödinger representation*. A deeper analysis also shows that this representation is irreducible, i.e. the only bounded operators commuting with all $U(f)$ and $V(g)$ are multiples of the identity.

Theorem 2.34 (Stone-von Neumann)**.** *The Schrödinger representation is the only irreducible representation of the Weyl relations over $\mathbb{R}^n$ up to unitary equivalence.*

Proof. See [RS80, Thm VII.I.14]. □

Remark 2.35 (Canonical commutation relations)**.** Assume self-adjoint operators Q_j, P_k are given such that

1. we can define $U(f)$, $V(g)$ as in Eq. (2.26),
2. these $U(f)$, $V(f)$ are representation of the Weyl relations and
3. they are irreducible, then we can conclude according to the Stone-von Neumann theorem that these Q_j, P_k are unitarily equivalent to the choice in Eq. (2.22).

In that case the relations in (2.23) are an easy consequence of the Weyl relations. We will rephrase this situation in the following by saying the Q_j, P_k satisfy the *Weyl form of the CCR*. If, however, we only know that the Q_j, P_k are self-adjoint and satisfy the relations in (2.23) we *can not* deduce the Weyl relations. All corresponding calculations you might find are formal and can not be made rigorous *without additional assumptions*. In general, (2.23) is not even suffcient to gurantee that $U(f)$ and $V(g)$ can be defined according to (2.26). As a consequence, there actually are examples for self-adjoint operators Q, P satisfying (2.23), which are *not* unitarily equivalent to the version in (2.22); cf. [RS80, Sec. VIII.5].

With this remark we can return to the time-zero fields. We define a corresponding representation of the Weyl relations – this time over $V = \mathscr{S}(\mathbb{R}^3)$ rather than $V = \mathbb{R}^n$ and ask for uniqueness. The depressing answer is given in the next theorem.

Theorem 2.36. *Let φ_m, π_m be the time-zero field and conjugate momentum of the free scalar field of mass m. Then*

$$U_m(f) = \exp(i\varphi_m(f)) \qquad V_m(g) = \exp(i\pi_m(g))$$

is an irreducible representation of the Weyl relations over $\mathscr{S}(\mathbb{R}^4, \mathbb{R})$. Further, (U_{m_1}, V_{m_1}) and (U_{m_2}, V_{m_2}) are unitarily inequivalent if the masses are different, so if $m_1 \neq m_2$.

Proof. [RS75, Thm X.46]. □

Remark 2.37 (Interpretation)**.** The theorem shows that we actually can consider the Weyl form of the CCR as a mandatory requirement for time-zero fields, but this does not lead to a unique choice. Even worse, the example for inequivalent representations given depend on the mass. But mass is a dynamical parameter since it labels different versions of the field equations. Hence the representations of the Weyl realtions we have to deal with in field theory *does contain dynamical information.*

Remark 2.38 (Time evolution again)**.** The failure of unitary equivalence has another striking consequence concerning time evolution. To explain what we've just lost, let us have another look at the finite degree of freedom case. Here we can look at the observables Q_j, P_k in the Heisenberg picture and time-dependent operators

$$Q_j(t) = \exp(-itH)Q_k \exp(itH), \quad P_k(t) = \exp(-itH)P_k \exp(itH),$$

where H is a self-adjoint operator and the Hamiltonian of the model. If we assume the other way round that time-dependent, self-adjoint operators $Q_j(t), P_k(t)$ are given which satisfy the Weyl form of the CCR at each instance of time, we can conclude from the Stone-von Neumann theorem that there is a unitary $T(t)$ satisfying $T(t)Q_j(0)T(t)^* = Q_j(t)$ and $T(t)P_k(0)T(t)^* = P_k(t)$. It is still unclear (without further knowledge) whether these $U(t)$ are given as $\exp(-itH)$ in terms of a Hamiltonian H, but at least we know that the time evolution is given by unitaries on the same Hilbert space.

The lack of uniqueness in the field theoretical case indicates therefore that we can not expect that time evolution is unitary. This leads to the urgent question whether we can preserve uniqueness of the Weyl relations at least on a more abstract level. The rest of this section is devoted to an answer of this question. As a preparation we need some material about C*-algebras.

Definition 2.39. *Consider the space $\mathcal{B}(\mathcal{H})$ of bounded operators on a (not necessarily separable) Hilbert space $\mathcal{H}$. A linear subspace $\mathcal{A}$ which is closed under products*[14] *and adjoints*[15] *is called a C*-algebra.*

Definition 2.40. *An invertible linear map $\alpha : \mathcal{A}_1 \to \mathcal{A}_2$ between two C*-algebras $\mathcal{A}_1$, $\mathcal{A}_2$ is called *-isomorphism if $\alpha(AB) = \alpha(A)\alpha(B)$ and $\alpha(A^*) = \alpha(A)^*$ holds for all $A, B \in \mathcal{A}_1$. If $\mathcal{A}_1 = \mathcal{A}_2 = \mathcal{A}$, the map α is called *-automorphism.*

Remark 2.41 (C*-algebras)**.** The central references for all statements about C*-algebras are the books of Bratteli and Robinson [BR12; BR02]. The following is a short list of additional remarks.

1. C*-algebras can be defined alternatively in an abstract way. We have chosen the explicit form as algebras of operators since it leads us more directly to the desired goal.
2. C*-algebras are in particular *-algebras. Hence all the corresponding material from Sec. 1.3 applies. This in particular concerns states and representations.

[14] $A, B \in \mathcal{A} \Rightarrow AB \in \mathcal{A}$
[15] $A \in \mathcal{A} \Rightarrow A^* \in \mathcal{A}$

3. Consider the GNS representation $(\mathcal{H}_\omega, D_\omega, \pi_\omega, \Omega_\omega)$ of a state ω of a C*-algebra $\mathcal{A}$.
4. In the definition of *-isomorphisms we haven't added any boundedness requirements, because this is not necessary. All *-isomorphisms α satisfy $\|\alpha(A)\| = \|A\|$ for all $A \in \mathcal{A}_1$. Hence they are automatically bounded ($\|\cdot\|$ here denotes the operator norm).

Definition 2.42. *Consider a real pre-Hilbert space $(\mathcal{K}, \langle\cdot\,,\cdot\rangle)$ which is also a locally convex space, and a representation $\{U, V\}$ of the Weyl relations. The smallest C*-algebra $\mathcal{A}$ containing the operators $U(f), V(g)$ for all $f, g \in \mathcal{K}$ is called the Weyl-algebra of $\{U, V\}$.*

Theorem 2.43. *The Weyl algebras $\mathcal{A}_1$, $\mathcal{A}_2$ of two representations $\{U_1, V_1\}$ and $\{U_2, V_2\}$ over the same space $(\mathcal{K}, \langle\cdot\,,\cdot\rangle)$ are *-isomorphic. More precisely, there is a *-isomorphism $\alpha : \mathcal{A}_1 \to \mathcal{A}_2$ with $\alpha\big(U_1(f)\big) = U_2(f)$ and $\alpha\big(V_1(g)\big) = V_2(g)$ for all $f, g \in \mathcal{K}$.*

Proof. [BR02, Theorem 5.2.8] □

Remark 2.44 (Interpretation)**.** Assume that by some construction we got time-zero fields φ_t and π_t which by all instances of time satisfy the Weyl form of the CCR, then by the previous theorem we get a family of *-isomorphisms $\alpha(t)$ with $\alpha_t\big(\exp(i\varphi_0(f))\big) = \exp(i\varphi_t(f))$ and similarly for π_t. These α_t take the role of the unitaries $T(t)$ from Remark 2.38, and therefore we can hope for a dynamical description of field theories in terms of automorphisms of C*-algebras, rather than unitaries on Hilbert spaces. Or to formulate this in another way: The Hilbert space structure which served us well in quantum mechanics is too rigid for field theory and has to be replaced by C*-algebras. This point of view is quite successful at least on the conceptual side; cf. the book of Haag [Haa96] for an in-depth discussion of this point of view. For constructive purposes, however, our problems are not completely solved. It might (and does) in particular happen that time-zero fields do not exist at all, since their existence is not a consequence of the Wightman axioms. For the most non-trivial example of an interacting model – quartic self interactions in 1+1 dimensions – the material developed so far is sufficient. We consider this case in the next chapter.

3 Interactive Fields

3.1 Naive Approach

For the first part of this chapter, we orient ourselves towards [RS75, p.233 ff.]. An "interacting field theory" is a field theory satisfying the Wightman axioms which has a nontrivial scattering theory. A natural way to construct such fields (cf. the discussion in the last section; in particular Remark 2.30) is to try to make a pertubation of some free theory, so

$$\varphi_t(x) = e^{-itH}\varphi(x)e^{itH} \qquad \pi_t(x) = e^{-itH}\pi(x)e^{itH}$$

for some Hamiltonian H. This works if H is self-adjoint but otherwise most likely breaks the Poincaré invariance.

In classical Lagrangian field theory the simplest Hamiltonians are of the form

$$H = H_0 + \underbrace{\lambda \int_{\mathbb{R}^3} F(\varphi(x))\, dx}_{=H_I}$$

where F is some function, say a polynomial with some coupling constant λ. Since we want the Hamiltonian to be bounded below, we expect that the polynomial is of even order with positive highest coefficient. The most simple non-trivial case then is $F(x) = x^4$, so

$$H = H_0 + \lambda \int_{\mathbb{R}^3} \varphi(x)^4\, dx.$$

This expression does not make sense unless we consider the Wick-ordered case *(see Chapter 2.3)*

$$H = H_0 + \lambda \int_{\mathbb{R}^3} :\varphi(x)^4:\, dx,$$

which is at least a quadratic form. Unfortunately, this quadratic form does not arise from an operator since formally computing $H\Omega$ "$=$" Ψ for $\Psi^{(n)} = 0$ if $n \neq 4$ gives

$$\Psi^{(4)}(k_1, \ldots, k_n) = \int_{\mathbb{R}^3} \frac{\exp\Big(-ix\sum_{i=1}^4 k_i\Big)}{\prod_{i=1}^4 (2\pi)^{3/2}(2\omega(k_i))^{1/2}}\, dx = \frac{\delta\Big(\sum_{i=1}^4 k_i\Big)}{(2\pi)^{9/2}\prod_{i=1}^4 (2\omega(k_i))^{1/2}}$$

which certainly is not in L_2, alone because of the δ-function. This we can try to fix by smearing with a test function, so

$$\int_{\mathbb{R}^3} \frac{g(x)\exp\Big(-ix\sum_{i=1}^4 k_i\Big)}{\prod_{i=1}^4 (2\pi)^{3/2}(2\omega(k_i))^{1/2}}\, dx = \frac{\hat{g}\Big(\sum_{i=1}^4 k_i\Big)}{(2\pi)^{9/2}\prod_{i=1}^4 (2\omega(k))^{1/2}},$$

but we still do not get an L^2 function because $\omega(k_i)$ grows too slowly at infinity.

3.2 The Cut-off Hamiltonian

In order to obtain an operator, we restrict ourselves to one space dimension. The discussion of Wightman quantum fields and the construction of the free field has to be adopted accordingly. This is, however, staightforward and (again) left as an exercise for the reader. We will try to sketch the idea of the construction of the $(\varphi^4)_2$ model from [GJ68; GJ70a; GJ70b] and accompanying papers.

We replace the quadratic form $\int_{\mathbb{R}} :\varphi(x)^4: dx$ by $\int_{\mathbb{R}} g(x) :\varphi(x)^4: dx$ where $g(x)$ is a real-valued function in $\mathrm{L}^2(\mathbb{R})$. So we have

$$H_g = H_0 + H_I(g) = \int_{\mathbb{R}} \omega(k) a_k^* a_k \, dk + \int_{\mathbb{R}} g(x) :\varphi(x)^4: \, dx - E_g$$

with domain

$$D(H_g) = D_{\mathscr{S}} = \{\Psi \in F_0 \,|\, \Psi^{(m)} \in \mathscr{S}(\mathbb{R}^{n \cdot m})\}.$$

Here the constant E_g, also called self energy of the vacuum, is chosen so that

$$0 = \inf\{\text{spectrum}\, H_g\}.$$

Further E_g is finite because of the spatial cut-off and because of the limitation to only one space dimension.

For the sake of locality, for g one often chooses some smooth function of compact support which equals one on a very large interval. Either way, the effect of g is to turn off the interaction for large values of x. Therefore, g is called the space cut-off and $H(g)$ is called the spatially cut-off Hamiltonian for the $(\varphi^4)_2$ field theory.

Theorem 3.1 ([RS75], Theorem X.62)**.** *The spatially cut-off Hamiltonian H_g is essentially self-adjoint for any $g \in \mathscr{S}(\mathbb{R}, \mathbb{R})$.*

Now the solution to the cut-off field equation

$$\frac{\partial^2 \Phi}{\partial t^2} - \frac{\partial^2 \Phi}{\partial x^2} + m^2 \Phi + 4\lambda g \Phi^3 = 0, \tag{3.1}$$

also compare (2.4), is given by the field at time t

$$\varphi_{t,g}(x) = \exp(-itH_g)\varphi(x)\exp(itH_g)$$

with corresponding momentum at time t

$$\pi_{t,g}(x) = \exp(-itH_g)\pi(x)\exp(itH_g).$$

Here $\varphi(x)$ and $\pi(x)$ are chosen to be the time-zero fields of the free field. Please keep this in mind, since this choice will cause considerable trouble below. Since translations affect the cut-off and therefore the field, this does not belong to a Wightman field because it violates Lorentz invariance. The simplest idea would be to take the limit $g \to 1$.

Unfortunately, this means $E_g \to \infty$ so our Hamiltonian does not converge in a proper sense.

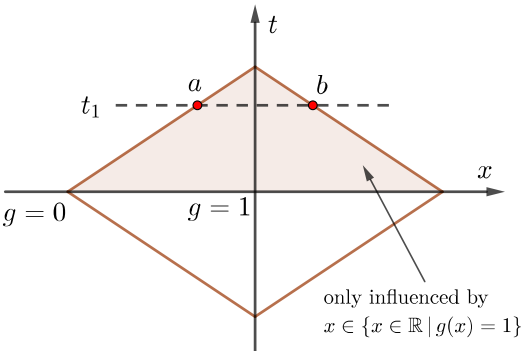


Figure 5: The field $\varphi_{t,g}(x)$ in two-dimensional space time. If $g(y) = 1$ for $|y| \leq |x| + |t|$, then the field is independent of g. This means for a time t_1, we get an associated interval (a, b) such that $\varphi_{t,g}(x) = \varphi_t(x)$ on the subset of the rhombus $(a, b) \times (-t_1, t_1)$.

However $\varphi_{t,g}(x)$ is independent of g provided $g(y) = 1$ for $|y| \leq |x| + |t|$ as is illustrated in Figure 5. This is a consequence of the fact that the field equation (3.1) propagates disturbances only with the speed of light (here set to 1), and therefore any changes to g made "far outside" can not influence anything in the rhombical region shown in the figure. Using this idea we can prove thr following

Theorem 3.2. *Consider $t_1 \in \mathbb{R}^+$ and $f_1, f_2 \in \mathscr{S}(\mathbb{R})$, with support in a bounded interval (a, b). Furthermore assume that $g \in \mathscr{S}(\mathbb{R}, \mathbb{R})$ satisfies $g(x) = 1$ for all $x \in (a-t_1, b+t_1)$. If we choose the time-zero fields $\varphi(f_1)$ and $\pi(f_2)$ of the free field we can define the quantities*

$$\begin{aligned} \varphi_t(f_1) &= \exp(-itH_g)\varphi(f_1)\exp(itH_g) \\ \pi_t(f_2) &= \exp(-itH_g)\pi(f_2)\exp(itH_g) \end{aligned} \tag{3.2}$$

for all $t \in [-t_1, t_1]$. They are self-adjoint operators (on an appropriate domain) and are independent of the cut-off g, provided it satisfies the given condition.

Proof. [GJ68, Sec. V] together with Lemma 3.2.2 and Theorem 3.2.6 of [GJ70a]. □

Hence, by choosing the interval, on which $g(x) = 1$ holds, large enough, we can define $\varphi_t(f_1)$ and $\pi_t(f_2)$ for all t and all $f_1, f_2 \in \mathscr{D}(\mathbb{R})$[16]. These fields have the cut-off removed.

From here on we can introduce spacetime fields (i.e. fields which are smeared over space and time rather than over space at a fixed time) by first defining

$$\Phi_g(f) = \int_{\mathbb{R}} \varphi_{t,g}(f_t)dt = \int_{\mathbb{R}} f(t,x)\varphi_g(t,x)dx\,dt \quad \text{with} \quad f \in \mathscr{D}(\mathbb{R}^2),\ f_t(\,\cdot\,) = f(t,\,\cdot\,),$$

then proving that this leads to a closable operator admitting an invariant dense domain [GJ70a, Theorem 3.2.3]. Using Theorem 3.2 we can show that this operator is independent of g provided the support of f is contained in the rhombical region from Figure 5; cf. [GJ70a, Thm 3.2.6]. In that way we get an operator-valued distribution on the Hilbert

[16] $\mathscr{D}(\mathbb{R}) = \{f \in C^\infty(\mathbb{R}) \,|\, \operatorname{supp} f \text{ compact}\}$

space $\mathcal{F}_+\big(\mathrm{L}^2(\mathbb{R})\big)$ with an invariant dense domain we haven't specified (cf. [GJ70a, Sect 3.2] for details; cf also [GJ70a, Thm 3.2.7] for the continuity requirements a distribution has to satisfy).

$$\mathscr{D}(\mathbb{R}^2) \ni f \mapsto \Phi(f) = \int_{\mathbb{R}^2} f(t,x)\varphi(t,x)dx\,dt, \tag{3.3}$$

which finally can be extended to test functions from $\mathscr{S}(\mathbb{R}^2)$, i.e. it is a *tempered* operator-valued distribution [GJ70a, Sec. 3.4]. We can recover the time-zero fields from Theorem 3.2 by restricting Φ and Φ_t to test "functions" δf; i.e.

$$\varphi_t(f_1) = \int_{\mathbb{R}} f_1(x)\Phi(t,x)dx \quad \text{and} \quad \pi_t(f_2) = \int_{\mathbb{R}} f_2(x)\partial_t\Phi(t,x)dx.$$

For the second equation cf. [GJ70a, Theorem 3.2.5]. Note that this shows in particular that the free field and the field Φ just constructed share the time-zero fields at $t = 0$. Furthermore it can be shown that this $\Phi(f)$ satisfies the correct field equations (Theorem 3.2.5 and the following remark of [GJ70a]), is self-adjoint if f is a real-valued function [GJ70a, Sec. 3.3], and is covariant with respect to spacetime translations [GJ70a, Sec. 3.6]. Furthermore, two operators $\Phi(f_1)$ and $\Phi(f_2)$ commute if the supports of f_1 and f_2 are spacelike separated [GJ70a, Sec. 3.5]. Hence we have almost constructed a Wightman field. The most important property we haven't shown yet (apart from covariance with respect to Lorentz boosts) is the existence of a vacuum, and actually, a vacuum vector $\Omega \in \mathcal{F}_+\big(\mathrm{L}^2(\mathbb{R})\big)$ *does not exist.*

3.3 Haag's Theorem

The fact that the above approach did not work out is not surprising but this was predicted by Haag's theorem which we will state now.

Theorem 3.3 ([Bog+90], Theorem 9.28)**.** *Let Φ, Φ_0 be two scalar Wightman fields acting on Hilbert spaces $\mathcal{H}, \mathcal{H}_0$. Assume for some instance of time t we have time-zero fields*

$$\Phi(t,x), \partial_t\Phi(t,x) \qquad \Phi_0(t,x), \partial\Phi_0(t,x)$$

which are irreducible and there exists unitary T such that

$$\begin{aligned} T\Phi(t,x)T^* &= \Phi_0(t,x) \\ T\partial_t\Phi(t,x)T^* &= \partial_t\Phi_0(t,x). \end{aligned} \tag{3.4}$$

Then all Wightman functions up to $n = 4$ are identical. If Φ_0 is a free field, then Φ is also free field in the sense that all Wightman functions are identical.

We can choose for Φ the field constructed in (3.3) and the free field for Φ_0. The method outlined in the last section used the time-zero fields for the free field to construct the interacting field Φ. Hence we have $\Phi(0,x) = \Phi_0(0,x)$ and $\partial\Phi(0,x) = \partial\Phi_0(0,x)$; cf. the definition of φ_t and π_t in Eq. (3.2) and check that $\pi_t(x) = \partial_t\Phi(t,x)$ really holds (in this context also cf. [GJ70a, Theorem 3.2.5]). Hence the condition in (3.4) is satisfied for $t = 0$

and $T = \mathbb{1}$. Since Φ_0 is the free field and Φ is not, the latter can not be a Wightman field. We are seeing here consequences of the non-uniqueness of representations of the Weyl relations, as discussed in Sec. 3.3.

3.4 Remove the Cut-off

The previous section clearly tells us that we have to come up with another construction to remove the cut-off. Although there is no vacuum vector for the field φ_t from (3.4), there is a vacuum vector Ω_g in $\mathcal{F}_+\big(\mathrm{L}^2(\mathbb{R})\big)$ for H_g:

Theorem 3.4. *For each cut-off g there is (up to scalar multiples) a unique vector $\Omega_g \in \mathcal{F}_+\big(\mathrm{L}^2(\mathbb{R}^3)\big)$ which is normalized ($\|\Omega_g\| = 1$) and satisfies $H_g\Omega_g = 0$.*

Proof. [GJ70a, Theorems 2.2.1 and 2.3.1] □

As partially stated before, simply taking the limit $g \to 1$ in the Fock space $\mathcal{F}_+\big(\mathrm{L}^2(\mathbb{R}^3)\big)$ yields the following problems.

- $E_g \to \infty$ so the Hamlitonian does not converge
- $\Omega_g \to 0$ in the weak sense

The way out of this conundrum is to follow the idea outlined in Remark 2.44 and to define the field dynamics algebraically, rather than in terms of unitaries. The first step is to define a C*-algebra $\mathcal{A}$ in terms of the fields $\Phi(f)$ from Eq. (3.3).

Definition 3.5. *With $\mathcal{H} = \mathcal{F}_+\big(\mathrm{L}^2(\mathbb{R}^2)\big)$ we define $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ as the smallest C*-algebra[17] containing all unitaries $\exp(i\Phi(f))$ for arbitrary $f \in \mathscr{D}(\mathbb{R}^2, \mathbb{R})$; i.e. smooth real-valued functions with compact support.*

Proposition 3.6. *There is a unique one parameter group α_t of automorphisms $\alpha_t : \mathcal{A} \to \mathcal{A}$ satisfying*

$$\alpha_t\big(\exp\big(i\Phi(f)\big)\big) = \exp(itH_g)\exp\big(i\Phi(f)\big)\exp(-itH_g),$$

where $g(x) = 1$ holds on a sufficiently large interval.

Proof. [GJ70a, Sec. 4]. □

This automorphism group describes the time evolution of the interacting fields algebraically. In a similar way we can proceed with the vacuum.

Definition 3.7. *For any $A \in \mathcal{A}$, we define the expectation values*

$$\omega_g(A) = \langle \Omega_g, A\Omega_g \rangle.$$

Theorem 3.8. *The following statements hold.*

(a) ω_g is a state of $\mathcal{A}$.

[17] We are glancing over some technicalities here, because the original construction in [GJ70b] is a bit more involved. However, the simplification we are applying here shouldn't cause big differences.

(b) There is a sequence $(g_n)_{n\in\mathbb{N}}$ *of cut-offs such that* $g \to \infty$ *and there is a state* ω *such that* $\lim_{n\to\infty} \omega_{g_n}(A) = \omega(A)$ *for all* $A \in \mathcal{A}$.

Proof. [GJ70b, Theorem 2.1]. □

Having this ω the idea is to get back a field which now should be a Wightman field. The idea is to use the GNS representation of the state ω just constructed. Hence we have a Hilbert space $\mathcal{H}_\omega$, a *-morphism $\pi_\omega : \mathcal{A} \to \mathcal{B}(\mathcal{H}_\omega)$, and a cyclic vector $\Omega_\omega \in \mathcal{H}_\omega$ such that $\omega(A) = \langle \Omega_\omega, \pi_\Omega(A)\Omega_\omega\rangle$.

Theorem 3.9. *There is a positive self-adjoint operator* H_ω *on* $\mathcal{H}_\omega$ *such that*

$$\exp(itH_\omega)\pi_\omega(A)\exp(-itH_\omega) = \pi\big(\alpha_t(A)\big) \quad \textit{and} \quad H_\omega\Omega_\omega = 0 \tag{3.5}$$

Proof. [GJ70b, Theorem 2.1]. □

Remark 3.10. (a) The first statement in (3.5) says that the automorphism group α_t is unitarily implemented in the representation π_ω, and the corresponding unitary group is generated by the operator H_ω which is the *renormalized Hamiltonian* of the interacting theory. It is the limit of H_g in the sense that

$$\langle \pi_w(B)\Omega_w, e^{iH_w t}\pi_w(A)\Omega\rangle$$

is obtained through a limit as $g \to 1$ of

$$\langle B\Omega_g, e^{iH_g t}A\Omega_g\rangle.$$

(b) The GNS vacuum Ω_ω is, again by Eq. (3.5) an eigenvalue of H with eigenvalue 0. Since H is positive, Ω_ω is eigenvector with the lowest energy.

(c) We call Ω_ω the physical vacuum and π_ω the physical representation. The vectors in $\mathcal{H}_\omega$ are called physical vectors.

Now let us come to the final step, which is the reconstruction of the fields in the new representation. This is done in terms of the following lemma.

Lemma 3.11. *Consider* $f \in \mathscr{S}(\mathbb{R}^2, \mathbb{R})$. *The one parameter unitary group*

$$\mathbb{R} \ni \lambda \mapsto V_\lambda = \pi\omega\big(\exp\big(i\lambda\Phi(f)\big)\big) \in \mathcal{U}(\mathcal{H}_\omega)$$

is strongly continuous.

Proof. This is a consequence of the "locally Fock" property proved in [GJ70b, Thm 2.2]; cf. also the corresponding remarks in Sec. 1 of [GJ70b]. □

Now we can introduce the *renormalized fields* as

$$\Phi_{\text{ren}}(f)\psi = \frac{1}{i}\frac{d}{d\lambda}V_\lambda\psi\Big|_{\lambda=0},$$

with the unitaries V_λ from the preceding lemma. The set of $\psi \in \mathcal{H}_\omega$ for which the limit exists forms the domain of $\Phi_{\text{ren}}(f)$. Now all properties of the field $\Phi(f)$ constructed in Sec. 3.2 can be carried over. To see this we again have to use the locally Fock property of π_ω from [GJ70b, Thm 2.2]. This basically finishes the construction. Let us summarize the result.

Theorem 3.12. *The fields $\Phi_{\text{ren}}(f)$ admit an invariant dense domain $D \subset \mathcal{H}_\omega$, and a strongly continuous repesentation $\mathcal{P}_+^\uparrow \ni (b, \Lambda) \mapsto U(b, \Lambda) \in \mathcal{U}(\mathcal{H}_\omega)$ such that the time translations coincide with $\exp(itH_\omega)$. The five-tuple $(\mathcal{H}_w, D, \Phi_\omega, \Omega_\omega, U)$ is a Wightman quantum field.*

Proof. This is not explicitly shown in [GJ70b] but implicitly discussed [GJ70b, Sec. 1]. Basically, all properties can either be carried over from the fields $\Phi(f)$ by using [GJ70b, Thm 2.2], or they are consequences of related properties of the GNS representation. The only missing point is the representation U. The space translations are treated in [GJ70b] along the same lines as the time evolution $\exp(itH_\omega)$. The whole Poincaré group is treated in [CJ70]. Together with the positivity of H_ω (which generates the time translations) this also shows that the spectrum condition holds. □

Remark 3.13. We again can use Haag's theorem to analyze this construction. Since the Wightman field we got is not the free field, by Theorem 3.3 said field has to be unitarily inequivalent to the free field for all times. Hence, the renormalization process just outlined, resulted in a change of representations of the CCR. The original one was the vacuum representation of the free field, which does not contain a Hamiltonian for the interacting model. The new representation is the vacuum representation of the interacting field and it does not contain the free Hamiltonian. A similar statement holds for the vacuum. The renormalized vacuum Ω_ω can not be described in terms of density operators in the vacuum representation of the free field. Roughly speaking, this is caused by the fact that we can say that this new vacuum contains infinitely many particles of the free theory, and therefore it is not defined in our original Fock space (which only contains states with finitely many particles).

4 Scattering Theory

Our goal is to present Haag-Ruelle scattering, which associates an S-matrix to fields. But in order to do so we need some preparations. Since I do not want to assume prior knowledge on this topic, we will start with some notes on classical particle scattering. This should motivate some of the constructions we will do in the following and simplifies the insight into the physical means. For this chapter we will orient ourselves towards [RS79].

4.1 Classical Particle Scattering

Phase space is given by

$$\Sigma = \mathbb{R}_q \times \mathbb{R}_p \qquad \Sigma_0 = \{(q,p) \in \Sigma \,|\, p \neq 0\}$$

and the free dynamics by

$$F_t^{(0)} : \Sigma \to \Sigma_0 \qquad (q,p) \mapsto \left(q + t\frac{p}{m}, p\right)$$

where m is the mass of the particle. Now for the interacting dynamics $K : \mathbb{R}^3 \to \mathbb{R}^3$ ("force") we assume boundedness and some kind of Lipschitz-continuity where the constant D_R depends on the radius of the considered disk.

$$\begin{aligned} \|K(q)\| &\leq C && \forall q \in \mathbb{R}^3 \\ \|K(q_1) - K(q_2)\| &\leq D_R \|q_1 - q_2\| && \forall q_1, q_2 \in \mathbb{R}^3 : \|q_1 - q_2\| \leq 1, \|q_{1/2}\| < R \end{aligned} \tag{4.1}$$

Now this implies that there exists a unique global solution of

$$\begin{aligned} \dot{q}(t) = \frac{p(t)}{m}, \dot{p}(t) = K(q(t)) \qquad t \in \mathbb{R} \\ q(0) = q, p(0) = p \qquad (q,p) \in \Sigma. \end{aligned} \tag{4.2}$$

Hence there exists a global flow

$$\begin{aligned} F : \mathbb{R} \times \Sigma &\to \Sigma \\ (t,q,p) &\mapsto (q(t), p(t)). \end{aligned}$$

The question now is if we can solve (4.2) with initial conditions "at infinity", i.e. by specifying asymptotic behaviour for $t \to \pm\infty$. For that we need further conditions on the force K.

$$\begin{aligned} \exists C > 0 \; \exists \alpha > 2 \; \forall q \in \mathbb{R}^3 \quad & \|K(q)\| \leq C\|q\|^\alpha \\ \exists D > 0 \; \exists \beta > 2 \; \forall r > 0 \; \forall \|q_{1/2}\| \geq r \quad & \|K(q_1) - K(q_2)\| < D r^{-\beta} \|q_1 - q_2\| \end{aligned} \tag{4.3}$$

Theorem 4.1. *Let K satisfy (4.1) and (4.3) and let $(q_{-\infty}, p_{-\infty}) \in \Sigma_0$ be given. Then*

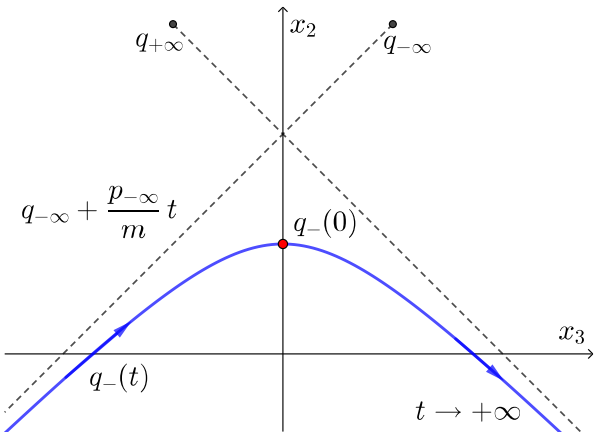
there exists a unique solution $\mathbb{R} \ni t \mapsto (q_-(t), p_-(t)) \in \Sigma$ *of (4.2) such that*

$$\lim_{t\to-\infty} \|p_-(t) - p_{-\infty}\| = 0$$
$$\lim_{t\to-\infty} \left\|q_-(t) - q_{-\infty} - \frac{p_{-\infty}}{m}t\right\| = 0$$

Similarly we get a solution $\mathbb{R} \ni t \mapsto (q_+(t), p_+(t)) \in \Sigma$ *if we replace* $-\infty$ *by* $+\infty$.

Proof. [RS79, Theorem XI.1] □

Figure 6: $p_{-\infty}$ is the initial momentum we start with at infinity. Similarly, $q_{-\infty}$ is the position of the *free* particle at time $t = 0$ which moves with momentum $p_{-\infty}$. Replace $-\infty$ with $+\infty$ to get the solution $t \mapsto (q_+(t), p_+(t))$ with asymptotics at $t \to +\infty$.



Definition 4.2. *Let* $t \mapsto (q_\pm(t), p_\pm(t))$ *be the solution from Theorem 4.1 which is asymptotic to* $t \mapsto q_{\pm\infty} + \frac{p_{\pm\infty}}{m}t$ *at* $t = \pm\infty$. *We define the Möller operators (wave operators)* $\Omega^\pm$ *via*

$$\Omega^\pm : \Sigma_0 \to \Sigma \quad (q_{\mp\infty}, p_{\mp\infty}) \mapsto (q_\mp(0), p_\mp(0)).$$

Ω^+ maps the $t = 0$ initial data of the $= -\infty$ asymptotic free dynamics to the $t = 0$ initial data of the interacting dynamics.

Proposition 4.3. *Let* K *satisfy (4.1) and (4.3). Then*

$$\Omega^\pm(q,p) = \lim_{t\to\mp\infty} F_{-t}F_t^{(0)}(q,p)$$

uniformly on compact subsets of Σ_0.

Proof. [RS79, Theorem XI.2.(a)] □

The $\Omega^\pm$ obviously are injective due to existence and uniqueness of the solutions, but not necessarily surjective.

Definition 4.4. *We define* $\Sigma_{in} = \operatorname{Ran}\Omega^+$, $\Sigma_{out} = \operatorname{Ran}\Omega^-$ *and*

$$\Sigma_{bound} = \{(q(t), p(t)) \,|\, \|q(t)\| + \|p(t)\| < \infty\}$$

where $\mathbb{R} \ni t \mapsto (q(t), p(t)) \in \Sigma$ *is the solution of (4.2) with* $(q,p) \in \Sigma$ *as initial data at* $t = 0$.

Definition 4.5. *We have weak asymptotic completeness if* $\Sigma_{in} = \Sigma_{out}$ *and asymptotic completeness if* $\Sigma_{in} = \Sigma_{out} = \Sigma \backslash \Sigma_{bound}$.

Note that all equalities hold up to measure zero sets.

Theorem 4.6. *Let* $K(q) = -\operatorname{grad} V(q)$ *with* $V(q) \to 0$ *if* $\|q\| \to \infty$. *If* K *satisfies (4.1) and (4.3) then we have asymptotic completeness.*

Proof. [RS79, Theorem XI.3] □

Now that we introduced the concept of asymptotic completeness we can define the S-matrix.

Definition 4.7. *Let* $\Sigma^{(\pm)} := (\Omega^{\pm})^{-1}(\Sigma \backslash \Sigma_{bound})$. *The S-matrix is the map*

$$S : \Sigma^{(+)} \to \Sigma^{(-)} \quad w \mapsto (\Omega^{-})^{-1}(\Omega^{+} w).$$

Note that we basically have to remove a measure zero set from Σ, i.e. $\Sigma' = \Sigma \backslash$measure zero set, and $\Sigma^{(\pm)} = (\Omega^{\pm})^{-1}(\Sigma')$.

As a special case we now consider a spherically symmetric potential $V(q) = V(\|q\|)$.

Proposition 4.8. *Let* $K(q) = -\operatorname{grad} V(q)$ *where* $V(q) = V(\|q\|)$. *Then the following statements hold.*

(1) $SF_t^{(0)} = F_t^{(0)} S$

(2) $SR = RS$ *for all* $R \in \mathrm{SO}(3)$ *where* $R(q,p) = (Rq, Rp)$

(3) (Energy conservation) $E_0(S(q,p)) = E_0(q,p) = \frac{1}{2m}\|p\|^2$

(4) (Angular momentum conservation) $L(S(q,p)) = L(q,p) = \frac{p}{m} \times q$

Proof. [RS79, Section XI.2] □

We use Proposition 4.8 to reduce the S-matrix to a pair of real valued functions of two real variables.

Step 1: The vectors p, q span a 2-plane. By (2) we can choose this to be the e_2, e_3-plane. We still can rotate around e_1 hence we choose $p = \|p\| e_3$.

Step 2: $S(q,p) = (q', p')$; now property (1) implies

$$S\Big(q + \frac{t}{m}p, p\Big) = \Big(q' + t\frac{p'}{m}, p'\Big).$$

Hence we can choose $q = be_2$. S is determined by its values on $(be_2, \|p\|e_3)$ for $b, \|p\| \in \mathbb{R}$. The quantity b is called *impact parameter*.

Step 3: By (3) we get $p' = \|p\| e(b, \|p\|)$ with some unit vector $e(b, \|p\|)$.

Step 4: By (4) the vectors (p', q') are in the e_2, e_3-plane and the component of p' which is perpendicular to $e(b, \|p\|)$ is determined by L.

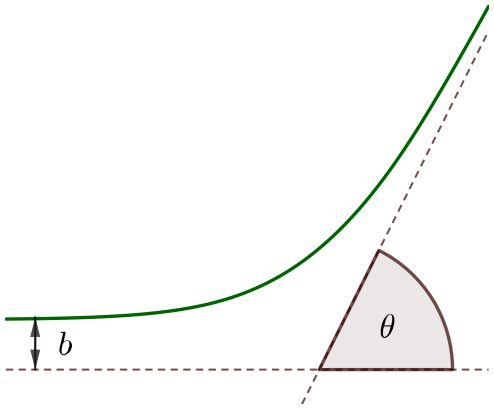
Thus, $S(q,p)$ is determined for all (q,p) by the two functions

$$\theta(b, \|p\|) = \arccos(e(b, \|p\|) \cdot e_3) \qquad \text{(Scattering angle)}$$

$$T(b, \|p\|) = \frac{q' \cdot e(b, \|p\|)m}{\|p\|} \qquad \text{(Time delay)}$$

with independant parameters b, $\|p\|$. The time delay measures the time the particle spent in the potential (compared to the free particle).

Figure 7: Schematic picture of central scattering. Here b is the impact parameter and θ is the scattering angle.



Remark 4.9 (Cross section)**.** Look at S and write $S(q,p) = (f(q,p), g(q,p))$ where we ignore f (e.g. ignore the time delay). Again we assume $p = \|p\|e_3$. By Proposition 4.8 (1) we get

$$g(q, \|p\|e_3) = g(q + \alpha e_3, \|p\|e_3)$$

for all $\alpha \in \mathbb{R}$. Hence it is sufficient to look at $q \in \mathbb{R}^3$ with $q \cdot e_3$. Energy conservation implies $\|g(q, \|p\|e_3)\| = \|p\|$ hence $\hat{g} = \frac{g}{\|p\|}$ takes values in S^2. With fixed $\|p\|$ we get

$$e_3^{\perp} \ni q \mapsto \hat{g}(q, \|p\|e_3) \in S^2.$$

The Lebesgue measure λ on $e_3^{\perp}$ induces an image measure on S^2 by

$$\sigma(E) = \lambda(\hat{g}^{-1}(E))$$

where $E \subset S^2$ Borel. If σ is absolutely continuous with respect to the invariant normalized measure Ω on S^2 we get a density

$$d\sigma = \frac{d\sigma}{d\Omega} d\Omega$$

where $\frac{d\sigma}{d\Omega}$ is the differential cross section.

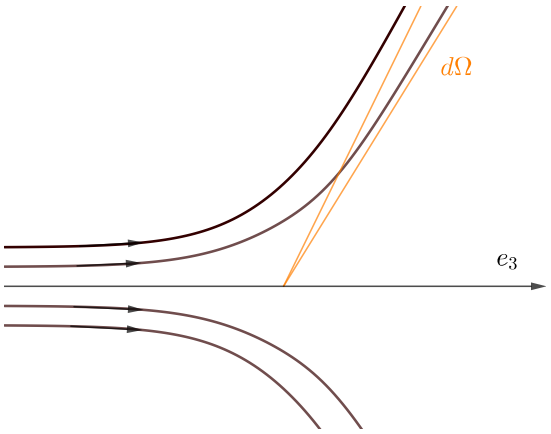


Figure 8: A uniform beam approaches in e_3-direction where the particles are scattered in angle elements $d\Omega$. Thus the differential cross-section can be measured in experiments.

The discussion just presented can not be applied to the Coulomb potential, since the time delay becomes infinite. This can be mitigated by a different choice of the free dynamics ("Dollard modifiers").

4.2 Quantum Scattering

Before diving into this section we repeat some basic concepts such as the spectrum of an operator *(Remark 4.10)* and types of operator convergence *(Remark 4.11)*.

Remark 4.10 (Spectrum)**.** Let H be an (unbounded) self-adjoint operator on some Hilbert space $\mathcal{H}$.

- $\lambda \in \mathbb{C}$ is in the resolvent set $\rho(H)$ if $\lambda \mathbb{1} - H$ is a bijection from $D(H)$ to $\mathcal{H}$ with bounded inverse.
- $\sigma(H) = \mathbb{C} \backslash \rho(H)$ is called the spectrum of H. For self-adjoint H we know that $\sigma(H) \subset \mathbb{R}$.
- For $f \in C^0(\sigma(A))$ vanishing at infinity (if A is unbounded) we define a bounded operator $f(A) \in \mathcal{B}(\mathcal{H})$ as follows. For $\Psi \in \mathcal{H}$ there is a unique measure μ_Ψ *(spectral measure)* on $\sigma(A)$ such that

$$\langle \Psi, f(A)\Psi \rangle = \int_{\sigma(A)} f(\lambda)\, \mu_\Psi(d\lambda).$$

- $\mathcal{H}$ decomposes as

$$\mathcal{H} = \mathcal{H}_{\text{pp}} \oplus \mathcal{H}_{\text{ac}} \oplus \mathcal{H}_{\text{sg}}$$

where

$$\begin{aligned}
\mathcal{H}_{\text{pp}} &= \{\Psi \,|\, \mu_\Psi \text{ is pure point}\} \\
\mathcal{H}_{\text{ac}} &= \{\Psi \,|\, \mu_\Psi \text{ is absolutely continuous w.r.t. the Lebesgue measure}\} \\
\mathcal{H}_{\text{sg}} &= \{\Psi \,|\, \mu_\Psi \text{ is singularly continuous}\}.
\end{aligned}$$

- For some $N \in \mathbb{N} \cup \{\infty\}$ we have

$$H|_{D(H)\cap\mathcal{H}_{\text{pp}}} = \sum_{j=1}^{N} \lambda_j |\phi_j\rangle\langle\phi_j|.$$

  If $N = \infty$ then this sum converges strongly. Elements of $\mathcal{H}_{\text{pp}}$ are bound states or linear combinations thereof.

- Let $\Psi \in \mathcal{H}_{\text{ac}}$. Then

$$\langle \Psi, H\Psi\rangle = \int_{\sigma(A)} \lambda \frac{d\mu_\Psi}{d\lambda}\, d\lambda$$

  where $\frac{d\mu_\Psi}{d\lambda}$ is the spectral density and $d\lambda$ is the Lebesgue measure. E.g. for the position operator we get

$$\langle \Psi, Q\Psi\rangle = \int_{\mathbb{R}} \overline{\Psi(x)} x \Psi(x)\, dx = \int_{\mathbb{R}} |\Psi(x)|^2 x\, dx.$$

  Elements of $\mathcal{H}_{\text{ac}}$ are unbound or scattering states. Up to a certain degree, scattering theory is about the analysis of $\mathcal{H}_{\text{ac}}$.

- With this we can define

$$\begin{aligned}
\sigma_{\text{pp}}(H) &= \{\lambda \,|\, \lambda \text{ is eigenvalue of } H\} \\
\sigma_{\text{ac}}(H) &= \sigma(H|_{\mathcal{H}_{\text{ac}}}) \\
\sigma_{\text{sg}}(H) &= \sigma(H|_{\mathcal{H}_{\text{sg}}}).
\end{aligned}$$

Note that we want to avoid the singularly continuous spectrum as it is not clear what it represents physically.

Remark 4.11 (Operator convergence)**.**

(1) (Norm convergence). We say that

$$\|\cdot\|\text{-}\lim_{N\to\infty} U_N = U$$

if $\|U_n - U\| \overset{N\to\infty}{\to} 0$. The good news is that if U_n unitary, then its norm limit U is unitary as well. The bad news however is that this type of convergence is too strong for quantum dynamics for the following reason. Let H be some *unbounded* self-adjoint opreator (e.g. the Hamiltonian of the harmonical oscillator) and $(t_n)_{n\in\mathbb{N}}$, $t_n \in \mathbb{R}$, $\lim_{n\to\infty} t_n = t$. Then

$$\|\cdot\|\text{-}\lim_{N\to\infty} \underbrace{\exp(it_nH)}_{=:V_n} \neq \exp(itH) =: V. \tag{4.4}$$

Furthermore for some complete orthonormal system $(\phi_n)_{n\in\mathbb{N}}$ in $\mathcal{H}$ we have

$$\|\cdot\|\text{-}\lim_{N\to\infty} \sum_{n=1}^{N} |\phi_n\rangle\langle\phi_n| \neq \mathbb{1}. \tag{4.5}$$

(2) (Weak convergence). This basically is the convergence of matrix elements as

$$\text{w-}\lim_{n\to\infty} U_n = U$$

means $\lim_{n\to\infty}\langle\phi, U_n\psi\rangle = \langle\phi, U\psi\rangle$ for all $\phi,\psi \in \mathcal{H}$. Now the good news is that (4.4) holds in the weak limit which is good for quantum dynamics and further, (4.5) converges weakly as well. However, $\text{w-}\lim_{n\to\infty} U_n$ for a sequence of unitary operators $(U_n)_n$ can now be any operator U with $\|U\| \leq 1$ [18] so the weak topology is too weak for our purposes (cf. Remark 4.12).

(3) (Strong convergence). The strong limit

$$\text{s-}\lim_{n\to\infty} U_n = U$$

is defined to be $\lim_{n\to\infty}\|U_n\phi - U\phi\| = 0$ for all $\phi \in \mathcal{H}$. Good news is that (4.4) and (4.5) converge strongly. At first it seems bad that $\text{s-}\lim_{n\to\infty} U_n = U$ for unitaries U_n only implies that U is an isometry[19]. The lack of unitarity of U is a consequence of the lack of strong continuity of the map $A \mapsto A^*$. However, we will see that lack of unitarity of the strong limit is exactly what we need in scattering theory. Hence strong topology is the correct choice for us.

Remark 4.12. Note that weak and strong topology induce the same topology on the

[18] As an example of this, consider $\mathcal{H} = \ell_2(\mathbb{N})$, $\mathcal{H}_{\text{a,b}} = \text{span}\{e_a,\ldots,e_b\} \subset \mathcal{H}$ and define $U_n e_j = e_{j+N}$ for $j = 1,\ldots,N$ and $U_N e_k = e_{k-N}$ for $k = N+1,\ldots,2N$. Hence $U_N\mathcal{H}_{1,N} = \mathcal{H}_{N+1,2N}$ and vice versa. Otherwise U_N resembles the identity $U_N e_l = e_l$ for all $l \in \{2N+1,\ldots\}$. Hence $\phi \in \mathcal{H}$ can be decomposed into $\phi = \phi_N \oplus \phi_{2N} \oplus \phi_R \in \mathcal{H}_{1,N} \oplus \mathcal{H}_{N+1,2N} \oplus \mathcal{H}_{2N+1,\infty}$ which yields

$$\begin{aligned}|\langle\phi, U_N\Psi\rangle| &= |\langle\phi_N, U_N\Psi_{2N}\rangle| + |\langle\phi_{2N}, U_N\Psi_N\rangle| + |\langle\phi_R,\Psi_R\rangle| \\ &\leq \|\phi\|\|U_N\Psi_{2N}\| + \|\Psi\|\|U_N^*\phi_{2N}\| + \|\phi_R\|\|\Psi_R\|\end{aligned}$$

as is readily verified. As $\phi,\Psi \in \ell_2$ they are square summable so for any $\varepsilon > 0$ there exists $N_\varepsilon \in \mathbb{N}$ such that $\|\Psi_{2N}\| + \|\Psi_R\| < \varepsilon$ and $\|\phi_{2N}\| + \|\phi_R\| < \varepsilon$ for all $N \geq N_\varepsilon$. Hence $\text{w-}\lim_{N\to\infty} U_n = 0$.

[19] This follows from

$$\langle\phi, U^*U\phi\rangle = \langle U\phi, U\phi\rangle = \|U\phi\|^2 = \lim_{N\to\infty}\|U_n\phi\|^2 = \|\phi\|^2.$$

About UU^* we can only say that it is a projection since

$$U\underbrace{U^*U}_{=\mathbb{1}}U^* = UU^* \qquad (UU^*)^* = U^{**}U^* = UU^*.$$

The operator U maps $\mathcal{H}$ entirely to the subspace $UU^*\mathcal{H}$.

unitary group $\mathcal{U}(\mathcal{H})$, i.e. for $(U_N)_{N\in\mathbb{N}}, U$ unitary we have

$$\operatorname*{s-lim}_{N\to\infty} U_N = U \;\Leftrightarrow\; \operatorname*{w-lim}_{N\to\infty} U_N = U$$

but they do not coincide on the unit ball and $\mathcal{U}(\mathcal{H})$ is neither strongly nor weakly closed in $\mathcal{B}(\mathcal{H})$.

Definition 4.13. *Let H, H_0 be self-adjoint operators on $\mathcal{H}$. The generalized wave operators $\Omega^{\pm}(H, H_0)$ exist if the strong limits*

$$\Omega^{\pm}(H, H_0) = \operatorname*{s-lim}_{t\to\mp\infty} e^{itH} e^{-itH_0} P_{ac}(H_0)$$

exist[20]*. If $\Omega^{\pm}(H, H_0)$ exist we define*

$$\mathcal{H}_{in} = \operatorname{Ran}\Omega^{+}(H, H_0) \qquad \mathcal{H}_{out} = \operatorname{Ran}\Omega^{-}(H, H_0).$$

This fits the formula from the classical case in Proposition 4.3.

Remark 4.14. (a) Here, H_0 is the "free" and H the "interacting" Hamiltonian where a typical choice is $H_0 = -\frac{\Delta}{2m}$. Hence in many (or even most) physically relevant cases H_0 only has absolutely continuous spectrum such that we have $P_{\mathrm{ac}}(H_0) = \mathbb{1}$ and

$$\Omega^{\pm}(H, H_0) = \operatorname*{s-lim}_{t\to\mp\infty} e^{itH} e^{-itH_0},$$

but this is by no means a requirement.

(b) If $P_{\mathrm{ac}}(H_0) = \mathbb{1}$ it can be shown that existence of $\|\cdot\|\text{-lim}\, e^{itH} e^{-itH_0}$ implies $H = H_0$ *(cf. [RS79, Section XI, Problem 15])*. Hence the norm topology is too strong as already stated before.

(c) (Partial isometries). An operator $U \in \mathcal{B}(\mathcal{H})$ is a partial isometry if $U : [E\mathcal{H}] \to [F\mathcal{H}]$ is unitary where $U^*U =: E$ is the initial and $UU^* =: F$ is the target projection.

Proposition 4.15. *Let $\Omega^{\pm}(H, H_0)$ exist. Then the following statements hold.*

(1) $\Omega^{\pm}(H, H_0)$ are partial isometries with $P_{ac}(H_0)$ as initial and $\mathcal{H}_{\substack{in\\out}}$ as target subspace.

(2) $\mathcal{H}_{\substack{in\\out}}$ are invariant subspaces for H and

$$\Omega^{\pm}(D(H_0)) \subset D(H) \qquad H\Omega^{\pm}(H, H_0) = \Omega^{\pm}(H, H_0)H_0.$$

(3) $\mathcal{H}_{\substack{in\\out}} \subset \operatorname{Ran} P_{ac}(H)$.

Proof. [RS79, Section XI.3, Proposition 1] □

[20] Here $P_{\mathrm{ac}}(H_0)$ denotes the projection onto the absolutely continuous spectrum of H_0.

Remark 4.16. (1) If $P_{\mathrm{ac}}(H_0) = \mathbb{1}$ then the operators $\Omega^{\pm}$ are isometries, i.e.

$$\big(\Omega^{\pm}(H, H_0)\big)^* \Omega^{\pm}(H, H_0) = P_{\mathrm{ac}}(H_0) = \mathbb{1}.$$

(2) From Proposition 4.15 (2), if $P_{\mathrm{ac}}(H_0) = \mathbb{1}$ we get

$$(\Omega^{\pm})^* H \Omega^{\pm} = H_0.$$

Hence restricting H to (parts of) its continuous spectrum leads to an operator which is unitarily equivalent to H_0.

(3) Obviously $\mathcal{H}_{\substack{\mathrm{in}\\ \mathrm{out}}} \subset \mathrm{Ran}(P_{\mathrm{ac}}(H))$ has to hold as $P_{\mathrm{pp}}(H)$ are the bound states and $P_{\mathrm{sg}}(H)$ we want to avoid.

(4) $\Omega^{\pm}$ maps the "free" states to the scattering states of H.

Definition 4.17. *Assume that $\Omega^{\pm}(H, H_0)$ exists. They are called complete iff*

$$\mathrm{Ran}\,\Omega^{+} = \mathrm{Ran}\,\Omega^{-} = \mathrm{Ran}\,P_{ac}(H).$$

If $P_{\mathrm{sg}}(H) = 0$ we get $\mathrm{Ran}\,\Omega^{+} = \mathrm{Ran}\,\Omega^{-} = \mathrm{Ran}\,P_{\mathrm{pp}}^{\perp}$ which is the equivalent of the condition in Definition 4.7 so we could again call this asymptotic completeness. In Definition 4.17 however, we haven't made any assumptions about the singular spectrum of H hence the condition there is only called completeness.

Definition 4.18. *Assume that $\Omega^{\pm}(H, H_0)$ exist and are complete. Then the S-matrix or scattering operator is defined to be*

$$S = (\Omega^{-})^* \Omega^{+}.$$

The following Proposition is the quantum analogon of Proposition 4.8.

Proposition 4.19. *The following statements hold.*

(1) $Se^{itH_0} = e^{itH_0}S$ for all $t \in \mathbb{R}$. The domain $D(H_0)$ is left invariant by S. Further if $\Psi \in D(H_0)$ then $H_0(S\Psi) = S(H_0\Psi)$.

(2) If $U \in \mathcal{U}(\mathcal{H})$ commutes with H, H_0 (i.e. $[U, e^{itH}] = 0$ etc.) then $[U, S] = 0$.

(3) S is unitary if and only if $\mathrm{Ran}\,\Omega^{+} = \mathrm{Ran}\,\Omega^{-}$ (weak asymptotic completeness).

Proof. [RS79, Proposition in Section XI.4, p.74] □

Remark 4.20. (a) There are two main tasks a mathematical physicist has to perform. One is to prove existence and completeness and two is determining S. For one there are lots of tools available. We do not have the time to go into details of [RS79]. For two there are two major strategies: (a) eigenfunction expansions and (b) pertubation theory. The former is treated a little bit in the next section. The latter is the most important tool in large parts of quantum field theory (in particular in QED). We will discuss it in detail in the next chapter.

(b) In some cases H_0 and H are not defined on the same Hilbert space (due to Haags theorem this happens in particular in Haag-Ruelle; cf. Sect. 4.4). This requires a special treatment known as two Hilbert space technique.

Definition 4.21. *Let H, H_0 self-adjoint operators on Hilbert spaces $\mathcal{H}, \mathcal{H}_0$ and $J \in \mathcal{B}(\mathcal{H}_0, \mathcal{H})$ be given. $\Omega^{\pm}(H, H_0; J)$ exist iff the strong limits*

$$\Omega^{\pm}(H, H_0; J) = \operatorname*{s-lim}_{t \to \mp\infty} e^{itH} J e^{-itH_0} P_{ac}(H_0)$$

exist.

Depending on J, the operators $\Omega^{\pm}(H, H_0; J)$ in general are no (partial) isometries. However, the following result holds.

Proposition 4.22. *$\mathcal{H}'_{in} := (\operatorname{Ker} \Omega^{+})^{\perp}$ is an invariant subspace for H_0 and $\mathcal{H}_{in} := \overline{\operatorname{Ran} \Omega^{+}}$ is an invariant subspace for H. Furthermore, $H_0|_{\mathcal{H}'_{in}}$ is unitarily equivalent to $H|_{\mathcal{H}_{in}}$. In particular $H|_{\mathcal{H}_{in}}$ is purely absolutely continuous.*

Proof. [RS79, Section XI.3, Proposition 4] □

Definition 4.23. *We say that $\Omega^{\pm}$ is semicomplete if $(\operatorname{Ker} \Omega^{\pm})^{\perp} = \operatorname{Ran} P_{ac}(H_0)$. If additionally $\overline{\operatorname{Ran} \Omega^{\pm}} = \operatorname{Ran} P_{ac}(H)$ then $\Omega^{\pm}$ is said to be complete.*

The whole construction up to a certain degree depends on the choice of J. There are criteria which guarantee that $\Omega(H, H_0; J_1) = \Omega(H, H_0; J_2)$, cf. [RS79, Section XI.3]. Physical input obviously is needed, too.

4.3 Quantum Two-Body Scattering

This is a short excursion, since the subject is not immediatly relevant to quantum field theory scattering. But it helps to relate the material to "ordinary" and "known" quantum mechanics. Let

$$\mathcal{H} = \mathrm{L}^2(\mathbb{R}^3) \otimes \mathrm{L}^2(\mathbb{R}^3) = \mathrm{L}^2(\mathbb{R}^6).$$

The free Hamiltonian is given by

$$\tilde{H}_0 = -\frac{1}{2\mu_1}\Delta_1 - \frac{1}{2\mu_2}\Delta_2$$

where $r \in \mathbb{R}^6$ is written $r = (r_1, r_2)$ with $r_i \in \mathbb{R}^3$ and Δ_i is the three-dimensional Laplacian associated to r_i. The interacting Hamiltonian is

$$\tilde{H} = \tilde{H}_0 + V(r_1 - r_2)$$

where V is a function in $\mathrm{L}^2(\mathbb{R}^3) + L^{\infty}(\mathbb{R}^3)$. Thus Kato's theorem *([RS75, Theorem X.16])* shows that $\tilde{H}$ is self-adjoint on $C_0^{\infty}(\mathbb{R}^6)$. We shall later place more severe restrictions on V.

First, we change coordinates to separate the center of mass motion. The new coordinates will be

$$R = \frac{\mu_1 r_1 + \mu_2 r_2}{\mu_1 + \mu_2} \qquad r_{12} = r_1 - r_2.$$

Let U be the unitary operator on $\mathrm{L}^2(\mathbb{R}^6)$ given by

$$(Uf)(x,y) = f\Big(\frac{\mu_1 x + \mu_2 y}{\mu_1 + \mu_2}, x - y\Big)$$

and let r_1, r_2 denote the obvious coordinate multiplication operators. Denote Ur_1U^{-1} by R and Ur_2U^{-1} by r_{12}. Then

$$\begin{aligned} U\tilde{H}U^{-1} &= -\frac{1}{2(\mu_1+\mu_2)}\Delta_R - \frac{1}{2m}\Delta_{r_{12}} + V(r_{12}) \\ U\tilde{H}_0U^{-1} &= -\frac{1}{2(\mu_1+\mu_2)}\Delta_R - \frac{1}{2m}\Delta_{r_{12}} \end{aligned}$$

where $m = \frac{\mu_1\mu_2}{\mu_1+\mu_2}$. We now write $\mathrm{L}^2(\mathbb{R}^6) = \mathrm{L}^2(\mathbb{R}^3) \otimes \mathrm{L}^2(\mathbb{R}^3)$ where the variables are R and r_{12}. Then, as operators on $\tilde{D} = C_0^\infty(\mathbb{R}^3) \otimes C_0^\infty(\mathbb{R}^3) \subset C_0^\infty(\mathbb{R}^6)$, $U\tilde{H}_0U^{-1}$ and $U\tilde{H}U^{-1}$ can be decomposed as

$$\begin{aligned} U\tilde{H}_0U^{-1} &= h_0 \otimes \mathbb{1} + \mathbb{1} \otimes H_0 \\ U\tilde{H}U^{-1} &= h_0 \otimes \mathbb{1} + \mathbb{1} \otimes H \end{aligned}$$

where

$$h_0 = -\frac{1}{2(\mu_1+\mu_2)}\Delta \qquad H_0 = -\frac{1}{2m}\Delta \qquad H = -\frac{1}{2m}\Delta + V(r).$$

Thus, $e^{-itU\tilde{H}_0U^{-1}} = e^{-ith_0} \otimes e^{-itH_0}$ and $e^{-itU\tilde{H}U^{-1}} = e^{-ith_0} \otimes e^{-itH}$. As these differ only in the second factor, we shall define wave operators $\Omega^\pm$ and a scattering operator S for the system $\{e^{-itH}, e^{-itH_0}\}$ on $\mathrm{L}^2(\mathbb{R}^3)$. The wave and scattering operators for the original system are then given by $U^{-1}(\mathbb{1} \otimes \Omega^\pm)U$ and $U^{-1}(\mathbb{1} \otimes S)U$ respectively.

The task now consists of first checking existence and completeness of the wave operators and second constructing the S-matrix.

Theorem 4.24 (Cook-Hack-Theorem)**.** *Let $V = V_2 + V_r \in \mathrm{L}^2(\mathbb{R}^3) + L^r(\mathbb{R}^3)$ for $2 \le r < 3$. Let $H_0 = -\Delta$ on $\mathrm{L}^2(\mathbb{R}^3)$ and $H = H_0 + V$. Then $\Omega^\pm(H, H_0)$ exist.*

Proof. [RS79, Theorem XI.24] □

Again the Coulomb potential is not covered. We have $(1 + \|v\|)^{-\beta} \in \mathrm{L}^2(\mathbb{R}^3) + L^r(\mathbb{R}^3)$ only if $\beta > 1$. As in the classical case we have to modify the free dynamics in order to handle an infinite time delay.

Theorem 4.25. *Let $H_0 = -\Delta$ on $\mathrm{L}^2(\mathbb{R}^3)$ and let $V(x) = V(\|x\|)$. Suppose that*

$$\int_1^\infty |V(r)|\,dr + \int_0^1 r|V(r)|\,dr < \infty.$$

Then $\Omega^\pm(H, H_0)$ exist and are complete.

Proof. [RS79, Theorem XI.31] □

Note that Coulomb again is not covered.

Remark 4.26 (Eigenfunction expansion, discrete spectrum)**.** An operator A on $\mathrm{L}^2(\mathbb{R}^3, dx)$ with purely discrete spectrum has an eigenfunction expansion in the direct sense that there are L^2-functions $\varphi_n(x)$ with an associated map $\tilde{}\colon \mathrm{L}^2(\mathbb{R}^3, dx) \to \ell_2$ by

$$(\tilde{f})_n = \int \overline{\varphi_n(x)} f(x)\,dx.$$

That the φ_n are eigenfunctions with $A\varphi_n = a_n\varphi_n$ can be expressed by

$$(\tilde{Af})_n = a_n \tilde{f}_n \qquad \text{if } f \in D(A).$$

The orthonormality of the $\{\varphi_n\}$ implies $\mathrm{Ran}\,\tilde{} = \ell_2$. The completeness of the φ_n is expressed by

$$f(x) = \mathrm{L}^2\text{-}\lim \sum_{n=0}^{\infty} \tilde{f}_n \varphi_n(x).$$

Finally, as a consequence of completeness and orthonormality we have $\|f\|^2 = \sum_n |\tilde{f}_n|^2$.

Remark 4.27 (Eigenfunction expansion, Laplacian)**.** We shall show how the Fourier transform provides an eigenfunction expansion for $H_0 = -\Delta$ which has only continuous spectrum. Write $\varphi_0(x, k) = e^{ik\cdot x}$ and think of $\varphi_0(\cdot, k)$ as a family of functions of x parametrized by a continuous index k. Then, we know that $\hat{}$ satisfies

$$\hat{f}(k) = \frac{1}{(2\pi)^{n/2}}\,\text{l.i.m.} \int \overline{\varphi_0(x,k)} f(x)\,dx$$

where $\text{l.i.m.} \int = \mathrm{L}^2\text{-}\lim \int_{|x|<M}$ as $M \to \infty$. The $\varphi_0(\cdot, k)$ are eigenfunctions with eigenvalue k^2 in the sense that

$$(\widehat{H_0 f})(k) = k^2 \hat{f}(k) \qquad \text{if } f \in D(H_0). \tag{4.6}$$

The orthogonality and "normalization" of the $\varphi_0(\cdot, k)$ imply $\mathrm{Ran}\,\hat{} = \mathrm{L}^2(\mathbb{R}^3, dx)$. The completeness of the set $\{\varphi_0(\cdot, k)\}_{k\in\mathbb{R}^n}$ is expressed by

$$f(x) = \text{l.i.m.}\, \frac{1}{(2\pi)^{3/2}} \int \varphi_0(x,k)\hat{f}(k)\,dx$$

and $\|f\|^2 = \int |\hat{f}(k)|^2\, dk$.

Remark 4.28 (Lippmann-Schwinger equation)**.** How can we find candidates φ for the "continuum eigenfunctions" needed for an eigenfunction expansion of $H = H_0 + V$? Remember that $\Omega^+ f$ has been defined only for $f \in \mathrm{L}^2$, but suppose that we could make sense out of $\Omega^+\varphi_0(\cdot, k)$. Then since $\Omega^+ H_0 = H\Omega^+$, $\varphi(\cdot, k) = \Omega^+\varphi_0(\cdot, k)$ should obey $H\varphi = k^2\varphi$ in the sense of (4.6). If $\varphi = \Omega^+\varphi_0$ in some sense, then $\varphi_0 = (\Omega^+)^*\varphi$ should be in the limit as $t \to -\infty$ of

$$\begin{aligned} e^{+iH_0t}e^{-iHt}\varphi &= \varphi - i\int_0^t e^{iH_0s}Ve^{-iHs}\varphi\, ds \\ &\to \varphi - \lim_{\varepsilon\downarrow 0} i\int_0^{-\infty} e^{iH_0s}Ve^{-ik^2s}e^{+\epsilon s}\varphi\, ds \\ &= \varphi + \lim_{\varepsilon\downarrow 0}(H_0 - k^2 - i\varepsilon)^{-1}V\varphi. \end{aligned}$$

Thus φ should obey

$$\varphi(\cdot, k) = \varphi_0(\cdot, k) - \lim_{\varepsilon\downarrow 0}([H_0 - (k^2 + i\varepsilon)]^{-1}V\varphi)(\cdot, k)$$

or, using the free resolvent

$$\varphi(x, k) = e^{ik\cdot x} - \frac{1}{4\pi}\int \frac{e^{ik|x-y|}}{|x-y|}V(y)\varphi(y, k)\, dy.$$

hence by solving the Lippmann-Schwinger equation we get "eigenfunctions" $\varphi(\cdot, k)$ with $\Omega^+\varphi_0(\cdot, k) = \varphi(\cdot, k)$. This directly gives us the Möller operators, and therefore the S-matrix. To make this formal reasoning exact look at [RS79, Theorem XI.41].

4.4 Haag-Ruelle Scattering Theory

The task is to associate an S-matrix to a Wightman quantum field. The first problem arising is the question what the free dynamics is. A possible solution to this, if the Hamiltonian of the interacting model has the form $H = H_0 + \lambda H_I$ (e.g. φ^4-model), is that we could use H_0. But we have seen that the free field for different masses leads to unitarily inequivalent models (cf. Section 3.4). It turns out that the mass m_0 belonging to H_0 is not the correct mass[21].

Assume that $(\mathcal{H}, D, A, \Psi_0, U)$ is a Wightman quantum field. To read off the correct mass from A we add the following axiom.

Definition 4.29 (Φ has an upper and lower mass gap)**.** *Let P_μ be the generators of the translation subgroup $U(a, I)$ of the Poincaré representation $U(a, \Lambda)$. For some $m > 0$ and*

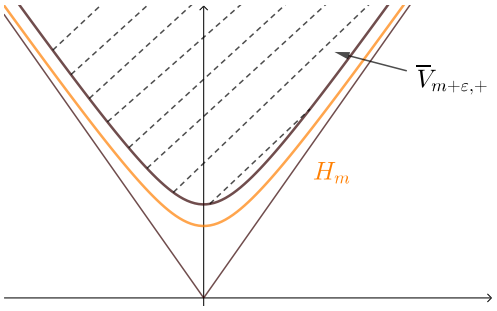
[21] The problem here is that bare mass is not the same as physical mass as a relativistic particle never escapes its own interaction.

some $\varepsilon > 0$, the spectrum of P_μ is contained in

$$\{0\} \cup H_m \cup \overline{V}_{m+\varepsilon,+} = \{0\} \cup \{p \,|\, p^2 = m^2; p_0 > 0\} \cup \{p \,|\, p^2 \geq (m+\varepsilon)^2; p_0 > 0\}$$

where $p^2 = p_0^2 - p_1^2 - p_2^2 - p_3^3$. Moreover, the set of vectors $\mathbb{S}$ which are eigenvectors for p^2 with eigenvalue m^2 is non-empty, and there is a cyclic vector for the action of $U(a, I)$ on $\mathbb{S}$.

Figure 9: Illustration of the condition in Definition 4.29.



Remark 4.30. $\mathbb{S}$ is the family of vectors describing the states of a single spinless particle of mass m. Definition 4.29 ensures that the eigenvalue m^2 is an isolated eigenvalue of P^2. Basically the idea is to create these one particle states by acting with a creation operator on the vacuum. To guarantee that this strategy works we need another property.

Definition 4.31 (Coupling of the vacuum to the one particle states)**.** *The spectral weight $d\rho$ for the Källen-Lehmann representation (see Theorem 2.26) has the form*

$$d\rho(s) = \delta(s - m) + d\tilde{\rho}(s)$$

where $d\tilde{\rho}$ has support in $[m + \varepsilon, \infty)$.

Remark 4.32. In the following we assume that A satisfies the conditions in Definitions 4.29 and 4.31. The basic idea to exploit these conditions is to use the mass from Def. 4.29 and consider the free field Φ_m of mass m. Now we want to define a map $J : \mathcal{F}_+(\mathrm{L}^2(H_m)) \to \mathcal{H}$ via

$$J\left[\prod_{i=1}^{n}(\Phi_m(g^{(i)}))\Omega_0\right] = \left[\prod_{i=1}^{n} A(g^{(i)})\right]\Psi_0 \tag{4.7}$$

with[22] appropriate test functions $g^{(i)}$. Unfortunately this does not lead to a well-defined quantity. Hence we need a little bit more preparation.

Definition 4.33. *A solution f of the Klein-Gordon equation (2.4) is called regular wave packet if the Fourier transform of the initial data $f(0,\cdot), \partial_t f(0,\cdot)$ are in C_0^∞.*

[22] Here, Φ_m is the free field with vacuum Ω_0 and A is the interacting field with vacuum Ψ_0.

Remark 4.34. From the analysis of Section 2.2 we see that $\phi = \phi_+ + \phi_-$ with

$$\phi_\pm(x,t) = \frac{1}{(\sqrt{2\pi})^{d-1}} \int e^{\pm i\omega(k)t} e^{ik\cdot x} u_\pm(k)\, d^{d-1}k$$

with $u_\pm(k) = \frac{1}{2}(\widehat{f_\pm(0,\cdot)} \mp \omega(k)^{-1}\widehat{\partial_t f_\pm(0,\cdot)})$. Hence $u_\pm \in C_0^\infty$.

Definition 4.35. *For two objects g, k (smooth functions or fields) we define the expression*

$$\left(g \overset{\leftrightarrow}{\partial_0} k\right)(t) = \int \left[g(x,t)\frac{\partial}{\partial t}k(x,t) - k(x,t)\frac{\partial}{\partial t}g(x,t)\right] d^3x.$$

Remark 4.36. (a) Consider the free scalar field Φ_m of mass $m > 0$ and a regular wave packet f. Now $\Phi_m(t,x)$ is given as a quadratic form

$$\int \Phi_m(t,x)[\varphi,\psi] g(t,x)\, dt\, dx = \langle \varphi, \Phi_m(g)\psi\rangle$$

for all $\varphi, \psi \in D_{\mathscr{S}}$ so $\varphi = \varphi^{(0)} \oplus \ldots \oplus \varphi^{(n)} \oplus 0 \oplus \ldots$ where $\varphi^{(j)} \in \mathscr{S}(\mathbb{R}^d \cdots \mathbb{R}^d)$. Recall that Φ_m and $\partial_t \Phi_m$ can be smeared over space alone

$$\Phi_{m,t}(g) = \int g(x)\Phi_m(t,x)\, dx \qquad (\partial_0 \Phi_{m,t})(g) = \int g(x)\partial_0\Phi_m(t,x)\, dx$$

so $\Phi_{m,t}, \partial_0\Phi_{m,t}$ are operators ("time zero fields"). Hence we can plug Φ_m into $g\overset{\leftrightarrow}{\partial_0}\Phi_m$. The latter is independant of t since f and Φ_m are solutions to the Klein-Gordon equation. If the Fourier transform of f in the spatial variables has the form

$$\hat{f}(p,t) = \frac{1}{(2\pi)^{1/2}} h(k) e^{-i\omega(k)(t)} \tag{4.8}$$

then we get a creation operator

$$f\overset{\leftrightarrow}{\partial_0}\Phi_m = i\int h(p)a^*(p)\, d^3p. \tag{4.9}$$

In particular, (4.9) shows that as n runs through $\mathbb{N}_0$ and f_i runs through all choices obeying (4.8), the vectors

$$\left(f_1 \overset{\leftrightarrow}{\partial_0}\Phi_m\right)\cdots\left(f_n\overset{\leftrightarrow}{\partial_0}\Phi_m\right)\Omega_0 \tag{4.10}$$

run through a total set of $\mathcal{H}_m = \mathcal{F}_+(\mathrm{L}^2(H_m))$, the Hilbert space of the free field. Thus for all $\psi \in \mathcal{H}_m$ there is *at most* one way to write ψ like so. Doing the opposite,

i.e.

$$\hat{f}(p,t) = \frac{1}{\sqrt{\omega}} \int h(p) e^{i\omega(p)t}\, d^3p$$

yields the annihilation operator

$$f \overset{\leftrightarrow}{\partial_0} \Phi_m = -i \int h(-p) a(p)\, d^3(p).$$

Hence this is a somewhat awkward way to recreate the Hilbert space of the free field.

(b) Now we can come back to the idea in (4.7) in Remark 4.32. We need an expression for the right-hand side of (4.7) similar to (4.10). Pick a function h in $C_0^\infty(\mathbb{R})$ so that $h(y) = 1$ near $y = m^2$ and $\operatorname{supp} h \subset (0, m^2 + \varepsilon)$. Define a new operator-valued distribution $B(x,t)$ by $\hat{B}(p) = h(p^2)\hat{A}(p)$ in momentum space, that is,

$$B(g) = A(Tg) \tag{4.11}$$

for any test function g where

$$\widehat{Tg}(p) = h(p^2)\hat{g}(p).$$

Now let $f \in \mathscr{S}(\mathbb{R}^3)$. Then $\hat{f}(p)e^{-ip_0t_0}h(p^2)$ is in $\mathscr{S}(\mathbb{R}^4)$, so we can pick g in (4.11) to have the form $f(x)\delta(t - t_0)$ with $f \in \mathscr{S}(\mathbb{R}^3)$. Hence $\hat{g}(p_0, p) = \hat{f}(p)e^{-ip_0t_0}$ so $\hat{g}(p_0,p)h(p^2) = \hat{f}(p)e^{-it_0p_0}h(p)$ is in $\mathscr{S}(\mathbb{R}^4)$. Thus $B(t,x)$ can be smeared with respect to space alone as $B(f,t) = A(T\,f \otimes \delta_t)$ is C^∞ in t. Now $B(\cdot,t)$ is smooth in t just as $\partial_t B(\cdot,t)$. In particular, for any $f \in C^\infty(\mathbb{R}^4)$ with $f(\cdot,t)$ and $\partial_0 f(\cdot,t)$ in $\mathscr{S}(\mathbb{R}^3)$ for each t, we can look at $(f \overset{\leftrightarrow}{\partial_0} B)(t)$ for a regular wave packet. By Definition 4.31

$$\left\langle \left(f \overset{\leftrightarrow}{\partial_0} B\right)\Psi_0, \left(g \overset{\leftrightarrow}{\partial_0} B\right)\Psi_0 \right\rangle = \left\langle \left(f \overset{\leftrightarrow}{\partial_0} B\right)\Omega_0, \left(g \overset{\leftrightarrow}{\partial_0} B\right)\Omega_0 \right\rangle \tag{4.12}$$

for the left-hand side of (4.12) can be written in terms of the two-point function for A. Since B has built into it a factor of $h(p^2)$, Definition 4.31 says that only the $\delta(s - m)$ term from the spectral weight survives. What is then left is the same thing that would occur if A was equal to Φ_m. Since $\hat{\Phi}_m(p) = h(p^2)\hat{\Phi}_m(p)$, (4.12) holds. In other words: B has the same two-point function as the free field. Hence the idea is to compare

$$J\left[\prod_{i=1}^n \left(f_i \overset{\leftrightarrow}{\partial_0} \Phi_m\right)\Omega_0\right] = \prod_{i=1}^n \left(f_i \overset{\leftrightarrow}{\partial_0} B\right)\Big|_{t=0} \Psi_0 \tag{4.13}$$

where $f_i \overset{\leftrightarrow}{\partial_0} B$ is the creation operator for one part of interacting theory. That this works is the content of the following theorem which is the main result of the Haag-Ruelle scattering theory.

Theorem 4.37. *Let A be a hermitian scalar Wightman quantum field, satisfying Definitions 4.29 and 4.31. Then the following statements hold.*

(a) For any regular wave packets $f_1, \ldots, f_n$, the limits

$$\lim_{t \to \mp\infty} \left(f_1 \overset{\leftrightarrow}{\partial_0} B\right) \cdots \left(f_n \overset{\leftrightarrow}{\partial_0} B\right)(t)\Psi_0 =: \eta_{\substack{in \\ out}}(f_1, \ldots, f_n)$$

exist in the norm topology on $\mathcal{H}$ and are independent of the choice of h.

(b) Let $\mathcal{H}_{in}$ and $\mathcal{H}_{out}$ denote the closed span[23] of the η_{in} and η_{out}. $\mathcal{H}_{in}$ and $\mathcal{H}_{out}$ are left invariant by the representation U of the Poincaré group.

(c) There exist operator-valued distributions φ_{in} on $\mathcal{H}_{in}$ and φ_{out} on $\mathcal{H}_{out}$ so that $(\mathcal{H}_{in}, U\varphi_{in})$ and $(\mathcal{H}_{out}, U\varphi_{out})$ are unitarily equivalent to the free fields of mass m and that

$$\eta_{\substack{in \\ out}}(f_1, \ldots, f_n) = \left(f_1 \overset{\leftrightarrow}{\partial_0} \varphi_{\substack{in \\ out}}\right) \cdots \left(f_n \overset{\leftrightarrow}{\partial_0} \varphi_{\substack{in \\ out}}\right)\Psi_0.$$

Proof. [RS79, Theorem XI.109] □

Remark 4.38. Let $f_1, \ldots, f_n$ obey (4.8). Now we can unambiguously define a map J from a dense subset of $\mathcal{H}_m$, the Hilbert space for the free field of mass m, to $\mathcal{H}$ by (4.13). Note that J is well-defined because a given vector $\psi \in \mathcal{H}_m$ can be written in at most one way as

$$\psi = \prod_{i=1}^{n} \left(f_i \overset{\leftrightarrow}{\partial_0} \Phi_m\right)\Omega_0$$

if the f_i are required to satisfy (4.8). Let $\mathscr{D}_0$ be the set of vectors $\prod_{i=1}^{n}(f_i \overset{\leftrightarrow}{\partial_0} \Phi_m)\Omega_0$ which is total in $\mathcal{H}_m$ by (4.9). Then the following result holds.

Corollary 4.39. *The limits*

$$\Omega^{\pm} = \operatorname*{s-lim}_{t \to \mp\infty} e^{itH} J e^{-itH_0}$$

exist and define partial isometries $\Omega^+ : \mathcal{H}_m \to \mathcal{H}_{in}$ and $\Omega^- : \mathcal{H}_m \to \mathcal{H}_{out}$.

Proof. Let $e^{-ith_0} f$ be defined via

$$(e^{-ith_0} f)(s, x) = f(s + t, x).$$

Then

$$e^{-itH_0}\left[\prod_{i=1}^{n} f_i \overset{\leftrightarrow}{\partial_0} \Phi_m\right]\Omega_0 = \left[\prod_{i=1}^{n} e^{-ith_0} f_i \overset{\leftrightarrow}{\partial_0} \Phi_m\right]\Omega_0$$

[23] The closed linear span of some non-empty subset $E \subseteq X$ of some normed vector space is the intersection of all the closed linear subspaces of X which contain E.

so

$$e^{itH}Je^{-itH_0}\left[\prod_{i=1}^{n} f_i \overset{\leftrightarrow}{\partial_0} \Phi_m\right]\Omega_0 = \prod_{i=1}^{n} e^{itH}\Big(f_i(\cdot,t)\overset{\leftrightarrow}{\partial_0} B(\cdot,0)\Big)e^{-itH}\Psi_0$$
$$= \prod_{i=1}^{n}\Big(f_i \overset{\leftrightarrow}{\partial_0} B\Big)(t)\Psi_0.$$

By Theorem 4.37, the limits defining $\Omega^{\pm}$ exist and

$$\Omega^{\pm}\left[\prod_{i=1}^{n} f_i \overset{\leftrightarrow}{\partial_0} \Phi_m\right]\Omega_0 = \prod_{i=1}^{n}\Big(f_i \overset{\leftrightarrow}{\partial_0} \varphi_{\substack{\text{in}\\\text{out}}}\Big)\Psi_0. \tag{4.14}$$

By (c) of said theorem, $\Omega^{\pm}$ are isometries. □

Remark 4.40. Eq. (4.14) says that

$$\Omega^{\pm}\Phi_m = \varphi_{\substack{\text{in}\\\text{out}}}\Omega^{\pm}.$$

In particular, the S-matrix $S = \Omega^{+}(\Omega^{-})^{*}$ obeys

$$\varphi_{\text{out}} = S^{-1}\varphi_{\text{in}}S$$

if we have the condition of asymptotic completeness

$$\mathcal{H}_{\text{in}} = \mathcal{H}_{\text{out}} = \mathcal{H}.$$

Since $\phi_{\text{in}}, \phi_{\text{out}}$ are free fields, they obey a particle interpretation in terms of the particle number operator. To understand this remark, note that as free fields ϕ_{in} and ϕ_{out} are accompanied by representations U_{in} and U_{out} of the Poincaré group, which are unitarily equivalent to the second quantization $\Gamma(U_m)$ of the irreducible representation U_m on $\mathrm{L}^2(H_m, \Omega_m)$ introduced in Prop. 2.3, the latter was identified as the description of a free, relativistic, elmentary particle of mass m and spin 0. Since $\mathcal{H}_{\text{in}}$ and $\mathcal{H}_{\text{out}}$ can be identified via the map J with the bosonic Fock space over $\mathrm{L}^2(H_m, \Omega_m)$, normalized vectors $\psi \in \mathcal{H}_{\text{in}}$ or $\psi \in \mathcal{H}_{\text{out}}$ can be regarded as (the description of) states of a finite number of mutually non-interacting particles of this type. They are non-interacting because they are located (at $t = \pm\infty$) at an infinite distance and therefore they do not recognize each other. This is possible, since the interaction mediated by massive particles is "short range", i.e. translated to the non-relativistic case: the interaction potential falls of faster than distance^{-1}. The S-matrix now describes how these free particles at $t = -\infty$ evolve into free particles at $t = +\infty$. In between, i.e. at finite times, the system interacts and a consistent interpretation in terms of particles is usually not possible.

5 Perturbative Theory à la Epstein-Glaser

5.1 Time-dependent Pertubation Theory in Quantum Mechanics

In this section we will have a short look at the time-dependent Schrödinger equation

$$\frac{d\varphi(t)}{dt} = -iH(t)\varphi(t)$$

and the corresponding scattering theory. For unbounded operators this equation is problematic, at least without further (and most likely very restrictive) statements about the domains of $H(t)$. Therefore we will assume that $H(t)$ is self-adjoint and bounded.

Definition 5.1. *A two-parameter family of unitary operators $U(s,t)$ for $s,t \in \mathbb{R}$ which satisfies the following conditions is called a unitary propagator.*

(a) $U(r,s)U(s,t) = U(r,t)$

(b) $U(t,t) = \mathbb{1}$

(c) $U(s,t)$ *is jointly strongly continuous in* s *and* t.

Theorem 5.2 (The Dyson expansion)**.** *Let $t \mapsto H(t)$ be a strongly continuous map of $\mathbb{R}$ into the bounded self-adjoint operators on a Hilbert space $\mathcal{H}$. Then*

$$U(t,s)\varphi = \varphi + \sum_{n=1}^{\infty}(-i)^n \int_s^t \int_s^{t_1} \dots \int_s^{t_{n-1}} H(t_1)\dots H(t_n)\varphi \, dt_n \dots dt_1$$

defines a unitary propagator on $\mathcal{H}$ such that for all $\Psi \in \mathcal{H}$ we have

$$\varphi_s(t) = U(t,s)\Psi$$

satisfies

$$\frac{d}{dt}\varphi_s(t) = -iH(t)\varphi_s(t) \qquad \varphi_s(s) = \Psi. \tag{5.1}$$

Proof. [RS75, Theorem X.69] □

Remark 5.3. We can motivate this setup by rewriting (5.1) as an integral equation

$$\varphi_s(t) = \varphi_s(s) - i\int_s^t H(t_1)\varphi_s(t_1)\, dt_1$$

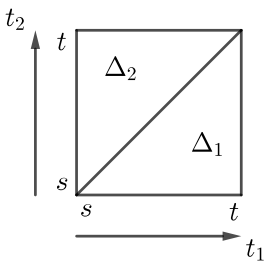
and inserting this expression recursively into itself. Of course this procedure does not prove anything.

Remark 5.4. We can extend the integrals to integrals over the whole interval $[s,t]$. Let's

have a look at $n = 2$.

$$\int_s^t \int_s^{t_1} H(t_1)H(t_2)\, dt_2\, dt_1$$

Figure 10: The domain of integration is a triangle. We obviously have $\Delta_2 = \{(t_1, t_2) \,|\, t_2 \geq t_1\}$ and $\Delta_1 = \{(t_1, t_2) \,|\, t_2 \leq t_1\}$.



Consider the map

$$F : (t_1, t_2) \mapsto (t_2, t_1) \qquad F(\Delta_2) = \Delta_1$$

so an integral over Δ_2 can be mapped to an integral over Δ_1 by flipping t_1 and t_2. Note that as $\det(DF) = 1$, the transformation of the integral does not lead to an additional factor. Fubini yields

$$\int_{\Delta_2} H(t_2)H(t_1)\, dt_2\, dt_1 = \int_{\Delta_1} H(t_1)H(t_2)\, dt_1\, dt_2 = \int_{\Delta_1} H(t_1)H(t_2)\, dt_2\, dt_1.$$

Now we introduce a "time-ordered product"

$$T\{H(t_1)H(t_2)\} = \begin{cases} H(t_1)H(t_2) & (t_1, t_2) \in \Delta_1 \\ H(t_2)H(t_1) & (t_1, t_2) \in \Delta_2 \end{cases}.$$

We then get

$$\int_s^t \int_s^{t_1} H(t_1)H(t_2)\, dt_2\, dt_1 = \int_{\Delta_1} H(t_1)H(t_2)\, dt_2\, dt_1 = \int_{\Delta_1} T\{H(t_1)H(t_2)\}\, dt_2\, dt_1$$

and

$$\int_s^t \int_s^{t_1} H(t_1)H(t_2)\, dt_2\, dt_1 = \int_{\Delta_2} H(t_2)H(t_1)\, dt_2\, dt_1 = \int_{\Delta_2} T\{H(t_1)H(t_2)\}\, dt_2\, dt_1.$$

For general n we define

$$T\{H(t_1) \dots H(t_n)\} = H(\sigma(t_1)) \dots H(\sigma(t_n))$$

where $\sigma : \{t_1, \dots, t_n\} \to \{t_1, \dots, t_n\}$ is a permutation with $\sigma(t_1) \geq \dots \geq \sigma(t_n)$. With an

induction argument we get

$$\int_s^t \int_s^{t_1} \dots \int_s^{t_{n-1}} H(t_1) \dots H(t_n)\, dt_n \dots dt_1 = \frac{1}{n!} \int_s^t \dots \int_s^t T\{H(t_1) \dots H(t_n)\}\, dt_n \dots dt_1.$$

Thus the Dyson series gets the more familiar form

$$\begin{aligned} U(t,s) &= \mathbb{1} + \sum_{n=1}^{\infty} \frac{(-i)^n}{n!} \int_s^t \dots \int_s^t T\{H(t_1) \dots H(t_n)\}\, dt_n \dots dt_1 \\ &=: T \exp\left(-i \int_s^t H(t_1)\, dt_1\right). \end{aligned}$$

Remark 5.5. By the proceeding remark we see that each term of the Dyson series is bounded by

$$\frac{|t-s|^k}{n!} \Big(\sup_{r \in [s,t]} \|H(s)\| \Big)^n \|\varphi\|.$$

Hence the series converges in norm to a unitary $U(s,t)$. Checking that $U(t,s)\Psi$ solves our original equation can now be done term by term. Checking further that $U(t,s)$ is a propagator is straightforward.

Remark 5.6. Although the Dyson expansion requires $H(t)$ to be bounded, by passing to the "interaction representation" we can use it to handle certain cases of the form

$$H(t) = H_0 + V(t)$$

where H_0 is a (possibly unbounded) self-adjoint operator and $t \mapsto V(t)$ satisfies the hypotheses of Theorem 5.2. Define

$$\tilde{V}(t) = e^{itH_0} V(t) e^{-itH_0}.$$

Then $t \mapsto \tilde{V}(t)$ also satisfies the hypotheses of Theorem 5.2 and we denote the corresponding propagator by $\tilde{U}(t,s)$. If we now set

$$U(t,s) = e^{-itH_0} \tilde{U}(t,s) e^{isH_0} \tag{5.2}$$

then, at least *formally*, $U(t,s)$ satisfies

$$\begin{aligned} \frac{d}{dt} U(t,s) &= -iH_0 e^{-itH_0} \tilde{U}(t,s) e^{isH_0} + e^{-isH_0} (-i\tilde{V}(t)) \tilde{U}(t,s) e^{isH_0} \\ &= (-iH_0 - iV(t)) U(t,s) \end{aligned} \tag{5.3}$$

so $\varphi_s(t) = U(t,s)\psi$ should be a strong solution of

$$\frac{d}{dt}\varphi_s(t) = -i(H_0 + V(t))\varphi_s(t) \qquad \varphi_s(s) = \psi.$$

The difficulty is that $H_0U(t,s)\psi = H_0e^{-itH_0}\tilde{U}(t,s)e^{isH_0}\psi$ may not make sense since $\tilde{U}(t,s)\psi$ may not be in the domain of H_0 even if ψ is. However, for any $\psi \in \mathcal{H}, \psi_s(t) = e^{-itH_0}\tilde{U}(t,s)e^{isH_0}\psi$ is always a "weak" solution in the sense that for any $\eta \in D(H_0)$, $\langle \eta, \psi_s(t)\rangle$ is differentiable and

$$i\frac{d}{dt}\langle \eta, \psi_s(t)\rangle = \langle H_0\eta, \psi_s(t)\rangle + \langle V(t)\eta, \psi, s(t)\rangle.$$

For our purposes the formal reasoning in (5.3) is sufficient. For a more detailed discussion of the validity of (5.3) and possible generalization we refer to [RS75, Section X.12].

Definition 5.7. *Let $U(t,s)$ be the unitary propagator associated to $H(t) = H_0 + V(t)$ according to Theorem 5.2. We say that the associated wave operators exist iff the strong limits*

$$\Omega^{\pm} = \operatorname*{s-lim}_{t\to\mp\infty} U(t,0)^*e^{-itH_0}$$

exist.

Remark 5.8. We can form the S-matrix as

$$S = (\Omega^-)^*\Omega^+ = \Big(\operatorname*{s-lim}_{t\to\infty} U(t,0)^*e^{-itH_0}\Big)^*\Big(\operatorname*{s-lim}_{s\to-\infty} U(s,0)^*e^{-isH_0}\Big).$$

If we ignore all warning signs and pretend that we can exchange adjoints with strong limits[24] and operator products[25] we get

$$S = \operatorname*{s-lim}_{\substack{t\to\infty\\ s\to-\infty}} e^{itH_0}U(t,0)U(s,0)^*e^{-isH_0}$$

with $U(s,0)U(0,s) = U(s,s) = \mathbb{1}$ and the other way round implies $U(0,s) = U(s,0)^*$ so we get

$$S = \operatorname*{s-lim}_{\substack{t\to\infty\\ s\to-\infty}} e^{itH_0}U(t,0)U(0,s)e^{-isH_0} = \operatorname*{s-lim}_{\substack{t\to\infty\\ s\to-\infty}} e^{itH_0}U(t,s)e^{-isH_0}.$$

However according to Remark 5.6 the operator

$$\tilde{U}(t,s) = e^{itH_0}U(t,s)e^{-isH_0}$$

[24] We can't as $A \to A^*$ is not strongly continuous.
[25] $(A,B) \mapsto AB$ is not jointly continuous in the strong operator topology.

is the unitary propagator belonging to the family

$$t \mapsto \tilde{V}(t) = e^{itH_0} V(t) e^{-itH_0}$$

which we can express as

$$\begin{aligned} S &= \operatorname*{s-lim}_{\substack{t\to\infty \\ s\to-\infty}} \tilde{U}(t,s) = \text{“}\tilde{U}(-\infty,-\infty)\text{”} \\ &= \mathbb{1} + \sum_{n=1}^{\infty} \frac{(-i)^n}{n!} \int_{\mathbb{R}^n} T\{\tilde{V}(t_1)\dots\tilde{V}(t_N)\}\, dt_1 \dots dt_n. \end{aligned} \tag{5.4}$$

This is the perturbation expansion of the S-matrix in quantum mechanics. Note that we have done several “forbidden” things like exchanging adjoints and strong limits. Hence we shouldn't be too surprised if the series does not converge, even in otherweise harmless situations. This is, however, not necessarily a problem. Even if convergence is not given, the first few (say two or three) terms of the expansion can for small potentials lead to reliable approximations of S. Cross sections derived from such a procedure can be reproduced with great accuracy in the laboratory[26].

5.2 Time-ordered Products

Let's have a look at the Hamiltonian of the Φ^4-model

$$H = H_0 + g \int_{\mathbb{R}^{d-1}} :\Phi_m(0,x)^4: \, d^{d-1}x \tag{5.5}$$

where $\Phi_m(t,x)$ is the free scalar field of mass $m > 0$ and H_0 is the corresponding Hamiltonian. Note that m is the “bare” mass, and not necessarily the “physical” mass identified in Section 4.4 via spectral properties of spacetime translations of the interacting fields. In fact the great advantage of this perturbative approach (or its biggest problem – this depends on your personal preferences) is that we do not need (and in most cases do not have) the fully interacting theory. The term

$$H_I = \int_{\mathbb{R}^{d-1}} :\Phi_m(0,x)^4: \, d^{d-1}x.$$

[26] This is the meaning of “reliable” in this context. We can look at it as in the Taylor expansion of a smooth but non-analytic function $f : \mathbb{R} \to \mathbb{R}$ around a value $x_0 \in \mathbb{R}$. Although the Taylor series does not converge, a Taylor polynomial $T_n f(x_0, x)$ for some order $n \in \mathbb{N}$ and small enough $|x_0 - x|$ can lead to a good approximation of $f(x)$. Mathematically the question is to understand what these approximations tell us and in what sense they are “good” approximations. For stationary pertubation theory such questions are studied in [RS78]. We are not following this route but continue with quantum field theory.

plays the role of the potential in the discussion of Remark 5.6. Hence we look at

$$\tilde{H}_I(t) = e^{iH_0t}H_Ie^{-iH_0t} = \int_{\mathbb{R}^{d-1}} :e^{iH_0t}\Phi_m(0,x)e^{-iH_0t}: d^{d-1}x = \int_{\mathbb{R}^{d-1}} :\Phi_m(t,x)^4: d^{d-1}x.$$

Using the formal expression (5.4) we get

$$S = \mathbb{1} + \sum_{n=1}^{\infty} \frac{(-i)^n g^n}{n!} \int_{\mathbb{R}^d} \dots \int_{\mathbb{R}^d} T\{:\Phi_m(x_1,t_1)^4: \dots :\Phi_m(x_n,t_n)^4:\} \, dx_n dt_n \dots dx_1 dt_1$$

or

$$S = \mathbb{1} + \sum_{n=1}^{\infty} \frac{(-i)^n g^n}{n!} \int_{\mathbb{R}^d} \dots \int_{\mathbb{R}^d} T\{:\Phi_m(y_1)^4: \dots :\Phi_m(y_n)^4:\} \, dy_n \dots dy_1 \tag{5.6}$$

with $y_j = (t_j, x_j)$. Again T denotes time ordering. This expression has several problems. First of all, the series does not converge to a unitary operator. After the discussion of the last section this is not a big surprise. Now, however, not even the terms for finite n are properly defined, and any effort to caclulate scattering amplitudes in a naive way by evaluating (5.6) formally leads to divergencies, either for small (infrared) or large momenta (ultraviolet). Based on our experiences from Chapter 3 where we have studied very simple interacting models we are not surprised about these difficulties either: to start out with free fields and treat the interacting model as a small perturbation thereof just does not work. In Chapter 3 we have had to change to a new representation of the CCR in order to get a mathematical well-defined quantum field. In the perturbative approach we have to find related methods to overcome the divergencies.

As a first step we replace the coupling constant g in (5.5) by a compactly supported, space and time (!) dependent smooth test function $g \in \mathscr{D}(\mathbb{R}^4)$. Hence we get

$$S(g) = \mathbb{1} + \sum_{n=1}^{\infty} \frac{(-i)^n}{n!} \int_{\mathbb{R}^d} \dots \int_{\mathbb{R}^d} g(y_1)\dots g(y_n) T\{:\Phi_m(y_1)^4: \dots :\Phi_m(y_n)^4:\} \, dy_n \dots dy_1. \tag{5.7}$$

This step removes all infrared divergencies from the model. To treat the latter we have to send the function g to a fixed value, and handle the corresponding limit of (5.7) appropriately, this is called the adiabatic limit. Some studies in this direction are already done be Epstein and Glaser [EG73], in these notes we skip this part. Instead we will have a short look at the local structure of the model, which is completely determined by knowing $S(g)$.

The second step needed to make sense of the formal expression in (5.6) is to define the time ordered product $T\{:\Phi_m(y_1)^4: \dots :\Phi_m(y_n)^4:\}$ as an operator-valued distribution for each fixed n . This can be done quite easily if all the spacetime events y_j are different. Poblems arise, however, on coincidence points, i.e. if $y_j = y_k$ holds for some $j \neq k$. To resolve these problems we follow a recursive procedure introduced by Epstein and Glaser [EG73]. The same method in the context of QED is descibed in detail in the book of

Scharf [Sch14]. In a curved spacetime context this method was first studied by Brunetti and Fredenhagen [BF00]. The latter is based on a slight modification of the orginial construction, which was proposed by Stora [Sto93]. We will loosely follow the paper of Brunetti and Fredenhagen, but translated back to Minkowski space.

Remark 5.9. Therefore we proceed with two complementary strategies.

(1) We look at $S(g)$ as a *formal* power series in g which is, however, well-defined term by term.

$$S(g) = \mathbb{1} + \sum_{n=1}^{\infty} \int \frac{1}{n!} T_n(y_1, \dots, y_n) g(y_1) \dots g(y_n)\, dy_n \dots dy_1$$

and for $f \in \mathscr{S}(\mathbb{R}^d)^n)$

$$f \mapsto \int f(y_1 \dots y_n) T_n(y_1, \dots, y_n) g(y_1) \dots g(y_n)\, dy_n \dots dy_1 = T_n(f)$$

is a well-defined operator-valued distribution on the Hilbert space $\mathcal{F}_+(\mathrm{L}^2(H_m)) =: \mathcal{H}$ of the free field. We do not care about convergence. In order to be able to define products $S(g_1) \dots S(g_n)$ we need $T_n(f)$ to be defined over the dense domain $D_{\mathscr{S}} \subset \mathcal{H}$ which is independent of n and f. Operator-valued distributions equal quantum fields but possibly without a cyclic vector.

(2) We characterize $S(g)$ axiomatically. In the case where $S(g)$ is given as a formal power series these axioms leads to conditions which we can use in the recursive construction just mentioned. We will start with some remarks concerning calculations with formal power series.

Remark 5.10 (Permutation invariance)**.** Since the products $g(y_1) \dots g(y_n)$ are permutation-invariant, the T_n can also be chosen to be permutation-invariant. For a set of arguments $X = \{y_1, \dots, y_n\}$ we will frequently write $T_n(X) = T_n(y_1, \dots, y_n)$. In terms of test functions this means that T_n is completely determined if its value on symmetric test functions is known. The space of the latter is denoted by $\mathscr{S}_{\mathrm{sym}}((\mathbb{R}^d)^n)$. In other words $f \in \mathscr{S}_{\mathrm{sym}}((\mathbb{R}^d)^n)$ iff $f \in \mathscr{S}((\mathbb{R}^d)^n)$ and $f(x_1, \dots, x_n) = f(x_{\pi(1)}, \dots, x_{\pi(n)})$ holds for all $(x_1, \dots, x_n) \in (R^d)^n$ and all permutations π. A general distribution can be projected to a permutation-invariant one by restricting it to symmetric test functions. We will frequently be doing this by first defining a non-symmetric one (e.g. as a tensor product) and then doing this projection (cf. the treatment of products of formal power series below). For later reference let us add the remark that the space of linear combintions of pure tensor powers of the form $g^{\otimes n} = g \otimes \dots \otimes g$ is dense in $\mathscr{S}_{\mathrm{sym}}((\mathbb{R}^d)^n)$, while the latter is a closed subpsace of $\mathscr{S}((\mathbb{R}^d)^n)$. Exercise: check both statements.

Remark 5.11. (a) (Formal power series, algebraic properties). Mathematically we can define a formal power series in g just as the seqeunce $(T_n)_{n\in\mathbb{N}}$ of operator-valued distributions. Writing them as a series is, however, a useful book keeping device

when algebraic operations are concerned. Look at

$$U(g) = \sum_{n=0}^{\infty} \frac{1}{n!} \int U_n(x_1, \ldots, x_n) g(x_1) \ldots g(x_n)\, dx_1 \ldots dx_n$$
$$V(g) = \sum_{n=0}^{\infty} \frac{1}{n!} \int V_n(x_1, \ldots, x_n) g(x_1) \ldots g(x_n)\, dx_1 \ldots dx_n$$

where $U_n(x_1, \ldots, x_n)$, $V_n(x_1, \ldots, x_n)$ are operator-valued distributions on $\mathcal{H}_m$ with domain $D_{\mathscr{S}}$. The linear combination for any $\lambda \in \mathbb{C}$ is given by

$$U(g) + \lambda V(g) = \sum_{n=0}^{\infty} \frac{1}{n!} \int (U_n(x_1, \ldots, x_n) + \lambda V_n(x_1, \ldots, x_n)) g(x_1) \ldots g(x_n)\, dx_1 \ldots .$$

Further the product is given by

$$W(g) = \sum_{n=0}^{\infty} \frac{1}{n!} \int W_n(x_1, \ldots, x_n) g(x_1) \ldots g(x_n)\, dx_1 \ldots dx_n$$

where

$$W_n(x_1, ..., x_n) = \sum_{m=0}^{n} \binom{n}{m} U_m(x_1, \ldots, x_m) V_{n-m}(x_{m+1}, \ldots x_n).$$

Note that this quantity in general is not permutation-invariant. We can solve this problem by a projection to the symmetric part (as described in Rem. 5.10), and (by using the same symbol in abuse of notation) get

$$W(X) = \sum_{I \subset X} U(I) V(X \backslash I).$$

Note that this is probably not an operator-valued distribution $\mathscr{S}(\mathbb{R}^{4n}) \to \mathrm{Op}(D_{\mathscr{S}}, \mathcal{H})$. The problem which might arise is that only for quantities of the form

$$\tilde{W}(f_1 \otimes f_2) = U_m(f_1) V_{n-m}(f_2), \quad f_1 \in \mathscr{S}((\mathbb{R}^d)^m),\ f_2 \in \mathscr{S}((\mathbb{R}^d)^{n-m})$$

we can be sure to get an operator $\tilde{W}(f_1 \otimes f_2)$. If $\psi, \phi \in D_{\mathscr{S}}$, we know by the nuclear theorem *(Theorem 1.13)* that $\tilde{W}_{\psi\phi}(f_1 \otimes f_2) = \langle \psi, W(f_1 \otimes f_n)\phi \rangle$ is a numerical distribution, i.e. $\tilde{W}_{\psi\phi}$ can be smeared by any test function $f \in \mathscr{S}((\mathbb{R}^d)^n)$. However, it is not clear whether the quadratic form given by $\tilde{W}(f)[\psi, \phi] = \tilde{W}_{\psi,\phi}(f)$ belongs to an operator. Note that we have dealt with a similar problem already in Prop. 1.40. The method used there is not applicable here. We will solve this difficulty by restricting the possible choices for the U_n, V_n.

(b) (Inverses). For expressions of the form $\mathbb{1}+T(g)$ we can define the inverse as follows.

$$(\mathbb{1}+T(g))^{-1}=\sum_{n=0}^{\infty}(-T(g))^n.$$

We get

$$\begin{aligned}(\mathbb{1}+T(g))\sum_{n=0}^{\infty}(-T(g))^n&=\sum_{n=0}^{\infty}(-T(g))^n+\sum_{n=0}^{\infty}T(g)(-T(g))^n\\&=\mathbb{1}+\sum_{n=1}^{\infty}(-T(g))^n-\sum_{n=0}^{\infty}(-T(g))^{n+1}=\mathbb{1}.\end{aligned}$$

We want that $(\mathbb{1}+T(g))^{-1}$ again is a formal power series which yields

$$(\mathbb{1}+T(g))^{-1}=\sum_{n=0}^{\infty}(-T(g))^n=\sum_{m=0}^{\infty}\frac{1}{m!}\int\tilde{T}_m(x_1,\ldots,x_m)g(x_1)\ldots g(x_m)\,dx_1\ldots dx_m \tag{5.8}$$

with

$$\tilde{T}_m(x_1,\ldots,x_m)=\sum_{n=0}^{m}(-1)^n\sum_{I_1,\ldots,I_n}T(I_1)\ldots T(I_n)$$

where the second sum on the right-hand side is taken over all partitions of $X=\{x_1,\ldots,x_m\}$ with $X=I_1\cup\ldots\cup I_n$, $I_j\cap I_k=\emptyset$ for all $j\neq k$ and $I_j\neq\emptyset$ for all j. This follows from expanding the powers of $-T$ in (5.8), e.g.

$$(-T)^0=\mathbb{1},\ (-T)^1=-T,\ (-T)^2=T^2=\int T_1(x_1)g(x_1)\,dx_1+\int T_1(x_2)g(x_2)\,dx_2+\ldots$$

Thus

$$(\mathbb{1}+T(g))^{-1}=\mathbb{1}-\underbrace{T_1(g)}_{=\tilde{T}_1(g)}-\underbrace{T_2(g\otimes g)+(T_1\otimes T_1)(g\otimes g)}_{=\tilde{T}_2(g\otimes g)}+\ldots$$

(from $-T$; from $-T$; from T^2)

etc[27].

(c) (Adjoints). Define

$$S(g)^*=\mathbb{1}+\sum_n\frac{1}{n!}\int\underbrace{T_n^+(x_1,\ldots,x_n)\overline{g(x_1)}\ldots\overline{g(x_n)}}_{=T_n^*(g\otimes\ldots\otimes g)|_{D_{\mathscr{S}}}}dx_1\ldots dx_n.$$

[27] Here, $(T_1\otimes T_1)(g\otimes g)$ is short for $\int T_1(x_1)T_1(x_2)g(x_1)g(x_2)\,dx_1dx_2$ to simplify the notation.

(d) Products with operators $A : D_{\mathscr{S}} \to D_{\mathscr{S}} \subset \mathcal{H}$ are given by

$$AS(g) = A + \sum_n AT_n(g \otimes \ldots \otimes g).$$

Thus, the set of formal power series becomes a complex $*$-algebra (provided we can handle the problem with products mentioned above).

Now we want to present the already mentioned axioms the S-matrix has to fulfill. In this context we can look at $S(g)$ either as formal power series on the free field Hilbert space $\mathcal{H}_m$ or as a unit operator on the Hilbert space of an interacting field $\mathcal{H}$ (possibly derived with methods similar to those discussed in Sect. 4.4).

Definition 5.12. *$S(g)$ is called*

(a) unitary if $S(g)^{-1} = S(g)^*$.

(b) translation invariant if

$$U(a, \mathbb{1})S(g)U(a, \mathbb{1})^{-1} = S(g_a) \qquad g_a(x) = g(x - a)$$

with $U : \mathcal{P}_+^\uparrow \to \mathcal{U}(\mathcal{H}_m)$ or $\mathcal{U}(\mathcal{H})$ being the representation of the Poincaré group associated to the free field ($\mathcal{H}_m$) or the interacting field ($\mathcal{H}$).

(c) Lorentz invariant if

$$U(0, \mathbb{1})S(g)U(0, \mathbb{1})^* = S(\Lambda g) \qquad \Lambda g(x) = g(\Lambda^{-1}x).$$

Definition 5.13. *For $x, y \in \mathbb{R}^4$ we write $x \gtrsim y$ if*

$$x \cap (y - \overline{V}^-) = \emptyset$$

where $\overline{V}^-$ denotes the closed past cone.

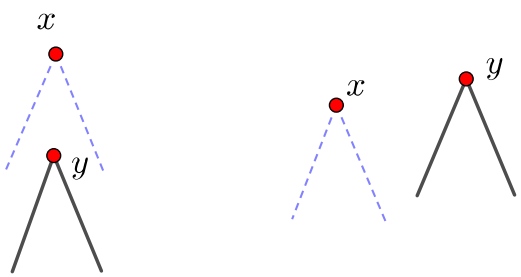


Figure 11: Illustration of Definition 5.13. In the left picture, y lies in the past cone of x but not vice versa so $x \gtrsim y$ but $y \not\gtrsim x$. Thus in the right picture, we obviously have $x \gtrsim y$ and $y \gtrsim x$.

Definition 5.14. *$S(g)$ is called causal if $S(g_1 + g_2) = S(g_1)S(g_2)$ whenever* $\operatorname{supp} g_1 \gtrsim \operatorname{supp} g_2$.

Remark 5.15. To motivate Definition 5.14 let's have a look at the quantum mechanical setting with a potential $V(t)$. Assume that we switch the interaction potential $V(t)$ on

and off with a cut off function $g : \mathbb{R} \to \mathbb{R}$. Hence we look at $g(t)V(t)$. For $g = g_1 + g_2$ with $g_1, g_2 \in C^\infty$ and $\operatorname{supp}(g_1) \subset (-\infty, s)$, $\operatorname{supp} g_2 \subset (s, \infty)$ we get

$$gV = g_1 V + g_2 V.$$

Formally we can calculate the S-matrix according to Eqs. (5.2) and (5.4). Hence with $U_0(t,s) = \exp(-i(t-s)H_0)$ we get

$$\begin{aligned} S(g_1+g_2) &= U_0(0,+\infty)\tilde{U}(+\infty,-\infty)U_0(-\infty,0) \\ &= U_0(0,+\infty)\tilde{U}(+\infty,s)U_0(s,0)U_0(0,s)\tilde{U}(s,-\infty)U_0(-\infty,0) = S(g_2)S(g_1) \end{aligned}$$

where the arguments $\pm\infty$ stand for the corresponding strong limits, and $\tilde{U}(t,s)$ is the propagator used in Eq. (5.2). Also note that we have used the fact that in the definition of $S(g_1)$ the interacting time evolution and the free time evolution between $-\infty$ and s coincide, such that we formally have

$$S(g_1) = U_0(0,+\infty)\tilde{U}(+\infty,-\infty)U_0(-\infty,0) = U_0(0,+\infty)\tilde{U}(+\infty,s)U_0(s,0).$$

Our next task is to translate the axioms into conditions for the T_n.

Proposition 5.16. *The following statements hold.*

(1) If $S(g)$ is unitary then $\tilde{T}_n = T_n^+$.

(2) If $S(g)$ is translation invariant then

$$U(a,\mathbb{1})T_n(x_1,\ldots,x_n)U(a,\mathbb{1})^* = T_n(x_1+a,\ldots,x_n+a)$$

when written with test functions.

(3) If $S(g)$ is Lorentz invariant then

$$U(a,\Lambda)T_n(x_1,\ldots,x_n)U(a,\Lambda)^* = T_n(\Lambda x_1,\ldots,\Lambda x_n).$$

(4) If S is causal, then $\{x_1,\ldots,x_m\} \gtrsim \{x_{m+1},\ldots,x_n\}$ implies

$$T_n(x_1,\ldots,x_n) = T_m(x_1,\ldots,x_m)T_{n-m}(x_{m+1},\ldots,x_n). \tag{5.9}$$

Remark 5.17. The formal expression used in Eq. (5.9) requires an explanation in term of test functions. Hence, consider open subsets $U_1 \subset (\mathbb{R}^d)^m$ and $U_2 \subset (\mathbb{R}^d)^{n-m}$ such that $\{x_1,\ldots,x_m\} \gtrsim \{x_{m+1},\ldots,x_n\}$ holds for all $(x_1,\ldots,x_m) \in U_1$ and all $(x_{m+1},\ldots,x_n) \in U_2$. Now we can choose test functions $f_1 \in \mathscr{S}_{\text{sym}}((\mathbb{R}^d)^m)$ and $f_2 \in \mathscr{S}_{\text{sym}}((\mathbb{R}^d)^{n-m})$. If $\operatorname{supp} f_j \subset U_j$, $j = 1,2$ holds, the operator-valued distribution should factorize as $T_n(f_1 \otimes f_2) = T_m(f_1)T_{n-m}(f_2)$. This should be regarded as the precise version of the formal statement in Eq. (5.9).

Proof. Only the last part requires a discussion. Consider open sets U_1, U_2 as in the previous remark and test functions $g_1, g_2 \in \mathscr{S}(\mathbb{R}^d)$ satisfying $\operatorname{supp} g_1^{\otimes m} \subset U_1$ and

$\operatorname{supp} g_2^{\otimes(n-m)} \subset U_2$. By assumption this implies $S(g_1+g_2) = S(g_1)S(g_2)$. Expanding S as a power series gives

$$\sum_{n=0}^{\infty}\sum_{m=0}^{n} \frac{1}{m!(n-m)!} \int T_n(x_1,\dots,x_n) g_1(x_1)\dots g_1(x_m) g_2(x_{m+1})\dots g_2(x_n)\, dx_1\dots dx_n$$
$$= S(g_1+g_2) = S(g_1)S(g_2) =$$
$$\sum_{n=0}^{\infty}\sum_{m=0}^{n} \frac{1}{m!(n-m)!} \int T_m(x_1,\dots,x_m) T_{n-m}(x_{m+1},\dots,x_n) g_1(x_1)\dots g_2(x_n) dx_1\dots dx_n$$

Comparing the two series term by term, leads to

$$T_n(g_1^{\otimes m} \otimes g_2^{\otimes(n-m)}) = T_m(g_1^{\otimes m}) T(g_2^{\otimes(n-m)})$$

Now the statement follows as linear combinations of tensor products $g_1^{\otimes m}$ are dense in $\mathscr{S}_{\mathrm{sym}}((\mathbb{R}^d)^m)$ and similarly for g_2; cf. Remark 5.10. □

5.3 Recursive Construction off the Diagonal

In this section we will construct the T_n by an inductive procedure on the space $\mathbb{R}^{dn}\backslash\Delta_n$ with $\Delta_n = \{(x,\dots,x)\,|\,x \in \mathbb{R}^d\}$. To this end it is convenient not to look only at $T(:\Phi_m(x_1)^4: \dots : \Phi_m(x_n)^4:)$ but to allow more general interaction Lagangians, i.e. $T(\mathcal{L}_1(x_1)\dots\mathcal{L}_n(x_n))$ with[28] $\mathcal{L}_j(x_j) =: \Phi_m(x_j)^{k_j}:$. A generalization to also include derivatives of the fields is straightforward but avoided here; cf. [EG73]. We start with a collection of properties the T_n should satisfy. They are mostly derived from Proposition 5.16.

Property 1. (Well-posedness). The symbols $T(\mathcal{L}_1(x_1)\dots\mathcal{L}_n(x_n))$ are well-defined operator-valued distributions on the Hilbert space $\mathcal{H}_m$ of the free field with invariant domain $D_{\mathscr{S}}$.

Property 2. (Symmetry). Any time-ordered product $T(\mathcal{L}_1(x_1)\dots\mathcal{L}_n(x_n))$ is symmetric under permutations of indices, i.e. the action of the permutation group of the index set $\{1,\dots,n\}$ gives (cf. Rem. 5.10)

$$T(\mathcal{L}_{\pi(1)}(x_{\pi(1)})\dots\mathcal{L}_{\pi(n)}(x_{\pi(n)})) = T(\mathcal{L}_1(x_1)\dots\mathcal{L}_n(x_n))$$

in the sense of distributions.

Property 3. (Causality). Consider any set of points $(x_1,\dots,x_n) \in (\mathbb{R}^d)^n$ and any full partition of the set $\{1,\dots,n\}$ into two non-empty subsets I and I^c such that no point x_i with $i \in I$ is in the past of the points x_j with $j \in I^c$, i.e. $x_i \notin J^-(x_j)$[29] for any $i \in I$ and $j \in I^c$. Then the time-ordered distributions are required to satisfy the following

[28] Here, we use the notation $T_n(x_1,\dots,x_n) = T(\mathcal{L}_{j_1}(x_{j_1}),\dots,\mathcal{L}_{j_n}(x_{j_n}))$ where the left hand side describes the time-ordered product and the $\mathcal{L}$ are the interaction Lagrangians.

[29] Here, $J^-(x_j) = x_j - \overline{V}_0$ where $\overline{V}_0 = \{v \in \mathbb{R}^p\,|\,g(u,v) \geq 0, v_0 \geq 0\}$ is the past cone. Thus $J^-(x_j)$ is the past and analogously, $J^+(x_j)$ the future causal cone at x_j.

factorization property (cf. Rem. 5.17)

$$T(\mathcal{L}_1(x_1)\dots\mathcal{L}_n(x_n)) = T\Big(\prod_{i\in I}\mathcal{L}_i(x_i)\Big)T\Big(\prod_{j\in I^c}\mathcal{L}_j(x_j)\Big)$$

Property 4. (Translation covariance).

$$T(\mathcal{L}_1(x_1+a)\dots\mathcal{L}_n(x_n+a)) = U(a,\mathbb{1})T(\mathcal{L}_1(x_1)\dots\mathcal{L}_n(x_n)U(a,\mathbb{1})^*$$

Remark 5.18. There is one important point missing, which addresses our concerns about products from Rem. 5.11. We are going to restrict the operator-valued distriutions $T(\mathcal{L}_1(x_1)\dots\mathcal{L}_n(x_n))$ to be within a certain class which is well-behaved under products. To this end we have to introduce some additional notations. As already stated we have $\mathcal{L}_j(x) =: \Phi_m(x)^{k_j}:$. Now for any integer $r \geq 0$ we can define

$$\begin{cases} \mathcal{L}_j^{(r)}(x) = \frac{k_j!}{(k_j-r)!} :\Phi_m(x)^{k_j-r}: & \text{for } k_j - r > 0 \\ \mathcal{L}_j^{(k_j)}(x) = k_j! & \\ \mathcal{L}_j^{(r)}(x) = 0 & \text{for } k_j - r < 0. \end{cases}$$

We can look at $\mathcal{L}^{(r)}(x)$ as a functional derivative of $\mathcal{L}(x)$ with respect to $\Phi_m(x)$. This point of view can be made rigorous by using commutators. Please check that $\mathcal{L}^{(1)}(x)$ satisifies and is uniquely determined by the conditon (cf. [BF00, Lemma 2.3])

$$\int f(x,y)[\mathcal{L}(x),\Phi_m(y)]\psi\,dx\,dy = \int f(x,y)E(x,y)\mathcal{L}^{(1)}(x)\psi\,dx\,dy \tag{5.10}$$

with $f \in \mathscr{S}(\mathbb{R}^d\times\mathbb{R}^d)$ and $\psi \in D_{\mathscr{S}}$. The distribution $E \in \mathscr{S}'(\mathbb{R}^d\times\mathbb{R}^d)$ is the "comutator function" given by

$$\int f(y)g(y)E(x,y)\,dx\,dy = \langle\Omega,\Phi_m(f)\Phi_m(g)\Omega\rangle.$$

Note that we have implicitly claimed that the product of the numerical distribution E and the operator-valued distribution $\mathcal{L}^{(1)}(x)$ on the right-hand side of (5.10) exists. Since E is translation invariant, this case is covered by the following theorem ([EG73, Theorem 0])

Theorem 5.19. *Let $F \in \mathscr{S}'(\mathbb{R}^{d\cdot n})$ be a tempered distribution such that $F(x_1,\dots,x_n) = F(x_1+a,\dots,x_n+a)$ for all $a\in\mathbb{R}^d$. Then, for any multi-index $r=(r_1,\dots,r_n)$ and any $f\in\mathscr{S}(\mathbb{R}^{d\cdot n})$*

$$\int F(x_1,\dots,x_n):\Phi_m(x_1)^{r_1}:\dots:\Phi_m(x_n)^{r_n}: f(x_1,\dots,x_n)\,dx_1\dots dx_n$$

is a well-defined operator on $D_{\mathscr{S}}\to D_{\mathscr{S}}\subset\mathcal{H}_m$. It depends continuously on f in the sense that the vector obtained by applying it to any vector of $D_{\mathscr{S}}$ depends continuously

on f in the norm topology of $\mathcal{H}_m$.

Remark 5.20. By looking at commutators and applying Eq. (5.10) we can reduce the order k_j of products $:\Phi_m(x_1)^{k_1}:\ldots:\Phi_m(x_n)^{k_n}:$. Combining this with an induction argument this leads to the generalized Wick expansion theorem (alternatively look at [Hep96])

$$\mathcal{L}_{j_1}(x_1)\ldots\mathcal{L}_{j_q}(x_q)=\sum_{s_1,\ldots,s_q}\langle\Omega,\mathcal{L}_{j_1}^{(s_1)}(x_1)\ldots L_{j_q}^{(s_q)}(x_q)\Omega\rangle\frac{:\Phi_m(x_1)^{s_1}\ldots\Phi_m(x_q)^{s_q}:}{s_1!\ldots s_q!}. \quad (5.11)$$

Hence the product of Wick monomials $:\Phi_m(x_j)^{k_j}:$ leads to a Wick *polynomial* with distributional coefficients. The latter are translation invariant such that we can apply Thm. 5.19 to see that each term on the right-hand side of (5.11) is an operator-valued distribution. Using Eq. (5.11) again, a straightforward calculation shows that the product $W(x_1,\ldots,x_n)=U(x_1,\ldots,x_k)V(x_{k+1},\ldots,x_n)$ of two such polynomials $U(x_1,\ldots,x_k)$, $V(x_{k+1},\ldots,x_n)$ is again a polynomial of the same type and therefore an operator-valued distribution $W(x_1,\ldots,x_n)$ on $(\mathbb{R}^d)^n$. In other words, as long as we can expand the $T(\mathcal{L}_{j_1}(x_1)\ldots\mathcal{L}_{j_q}(x_q))$ into Wick polynomials as just discussed, all the products of distributions we encounter are well-defined as (operator-valued) distributions and all the concerns from Rem. 5.11 are resolved. Based on that observation we add the condition

Property 5. (Causal Wick Expansion).

$$T(\mathcal{L}_1(x_1)\ldots\mathcal{L}_n(x_n))=\sum_{j_1,\ldots,j_n}\langle\Omega_\omega,T(\mathcal{L}_1^{(j_1)}(x_1)\ldots\mathcal{L}_n^{(j_n)}(x_n))\Omega_\omega\rangle\frac{:\Phi_m^{j_1}(x_1)\ldots\Phi_m^{j_n}(x_n):}{j_1!\ldots j_n!}$$

Remark 5.21 (Induction step)**.** We start the induction by setting $T(\mathbb{1})=\mathbb{1}$ and $T(\mathcal{L})=\mathcal{L}$ and assume that the time-ordered products for $1<l\leq n-1$ factors have been constructed and satisfy all the defining properties. In a first step, we define $T(\mathcal{L}_1(x_1)\ldots\mathcal{L}_n(x_n))$ on $\mathbb{R}^{d\cdot n}\backslash\Delta_n$ where Δ_n again is the set of coincidence points.

Remark 5.22. The basic idea for the definition of $T(\mathcal{L}_1(x_1)\ldots\mathcal{L}_n(x_n))$ is to use the expression

$$\tilde{T}(x_1,\ldots,x_n)=T(\mathcal{L}_1(x_1)\ldots\mathcal{L}_k(x_k))T(\mathcal{L}_{k+1}(x_{k+1})\ldots\mathcal{L}_n(x_n)).$$

Together with Property 5, the discussion in Rem. 5.20 shows that products of this form are operator-valued distributions on $(\mathbb{R}^d)^n$, and by the induction assumption both factors on the right-hand side of this equation are known. On the whole space, $\tilde{T}$ does not define the correct time ordered products (since in general we do not have $x_j\gtrsim x_\ell$ for all $j=1,\ldots k$ and all $\ell=k+1,\ldots n$). But we can restrict $\tilde{T}$ to the set

$$\mathcal{C}=\{(x_1,\ldots,x_n)\in(\mathbb{R}^d)^n\,|\,x_j\gtrsim x_\ell\ \forall j=1,\ldots,k\ \forall\ell=k+1,\ldots,n\} \quad (5.12)$$

and here the ordering is correct. Since $\mathcal{C}\subset(\mathbb{R}^d)^n$ is open this restriction defines an operator-valued distribution and our construction is complete – at least on $\mathcal{C}$. All we

have to do is apply the same idea to all sets arising from $\mathcal{C}$ by permuting the indices $j = 1, \ldots, n$, and look whether these local solutions fit together. To proceed in that direction, we need the following definition. Note that from here on we will follow [BF00] very closely.

Definition 5.23. *Let $\mathcal{J}$ be the set of all non-empty proper subsets I of $\{1, \ldots, n\}$. For each $I \in \mathcal{J}$ we define in analogy to (5.12)*

$$\mathcal{C}_I = \{(x_1, \ldots, x_n) \in (\mathbb{R}^d)^n \,|\, x_i \notin J^-(x_j), i \in I, j \in I^c\}.$$

Lemma 5.24. *The $\mathcal{C}_I$ overlap $(\mathbb{R}^d)^n \backslash \Delta_n$. In other words we have*

$$\bigcup_{I \in \mathcal{J}} \mathcal{C}_I = (\mathbb{R}^d)^n \backslash \Delta_n.$$

Proof. The inclusion $\cup_I \mathcal{C}_I \subseteq (\mathbb{R}^d)^n \backslash \Delta_n$ is obvious. The opposite inclusion is proved as follows. Consider any set of points $(x_1, \ldots, x_n)$ such that $x_i \neq x_j$ for some $i \neq j$. This implies that either x_i is in the causal future of x_j, or x_i is in causal past of x_j, or both are spacelike separated. In all cases we can find a spacelike hyperplane Σ such that one point is in future of Σ and the othe in the past, while none of the x_k, $k = 1, \ldots, n$ is an element of Σ. With slight loss of generality we assume that x_i is in the future of Σ (the other case is easily adopted). Now we define $I = \{k \,|\, x_k \text{ in the future of } \Sigma\}$. I is non-empty since $i \in I$ and it does not coincide with J since $j \notin I$. Hence $I \in \mathcal{J}$ and $(x_1, \ldots, x_n) \in \mathcal{C}_I$. □

Remark 5.25. We use the short hand notations

$$T^I(x_I) = T\Big(\prod_{i \in I} \mathcal{L}_i(x_i)\Big) \qquad x_I = (x_i, i \in I).$$

The first step now is to set on any $\mathcal{C}_I$

$$T_I(x) := T^I(x_I) T^{I^c}(x_{I^c})$$

as an operator-valued distribution. Recall from Rem. 5.22 that T_I is an operator-valued distribution on the whole space $(\mathbb{R}^d)^n$ and that its restriction has the correct operator ordering on the open subset $\mathcal{C}_I$ of $(\mathbb{R}^d)^n$. We now glue together all operators T_I for different $I \in \mathcal{J}$. To this end we have to show that on the overlaps $\mathcal{C}_{I_1} \cap \mathcal{C}_{I_2}$ the (restrictions of) the corresponding distributions T_{I_1} and T_{I_2} coincide. This is done in the following proposition (Prop. 4.2 of [BF00]).

Proposition 5.26. *For any choice of $I_1, I_2 \in \mathcal{J}$ such that $\mathcal{C}_{I_1} \cap \mathcal{C}_{I_2} \neq \emptyset$ we have*

$$T_{I_1}|_{\mathcal{C}_{I_1} \cap \mathcal{C}_{I_2}} = T_{I_2}|_{\mathcal{C}_{I_1} \cap \mathcal{C}_{I_2}}$$

in the sense of operator-valued distributions over $(\mathbb{R}^d)^n \backslash \Delta_n$.

Proof. Let $I_1, I_2 \in \mathcal{J}$ and $x = (x_1, \ldots, x_n) \in \mathcal{C}_{I_1} \cap \mathcal{C}_{I_2}$. Using the causality property *(Property 3)* which by assumption is valid for time-ordered products of less than n factors we find

$$\begin{aligned} T^{I_1}(x_{I_1}) &= T^{I_1 \cap I_2}(x_{I_1} \cap x_{I_2}) T^{I_1 \cap I_2^c}(x_{I_1 \cap I_2^c}) \\ T^{I_1^c}(x_{I_1^c}) &= T^{I_1^c \cap I_2}(x_{I_1^c} \cap x_{I_2}) T^{I_1^c \cap I_2^c}(x_{I_1^c \cap I_2^c}) \end{aligned} \tag{5.13}$$

and similarly for T^{I_2} and $T^{I_2^c}$. Now note that $(x_1, \ldots, x_n) \in \mathcal{C}_{I_1} \cap \mathcal{C}_{I_2}$ together with the definition of $\mathcal{C}_I$ implies that x_i with $i \in I_1 \cap I_2^c$ are neither in the causal past not in the causal future of x_j with $j \in I_1^c \cap I_2$. In other words they are spacelike separated, and therefore $T^{I_1 \cap I_2^c}$ and $T^{I_1^c \cap I_2}$ commute. This follows because we can write both terms by Property 5 (causal Wick expansion) as Wick polynomials, and Wick powers of the free field mutually commute at spacelike distances. Hence using (5.13) and Property 3, on $\mathcal{C}_{I_1} \cap \mathcal{C}_{I_2}$ we get

$$T_{I_1} = T^{I_1 \cap I_2} T^{I_1^c \cap I_2} T^{I_1 \cap I_2^c} T^{I_1^c \cap I_2^c} = T^{I_2} T^{I_2^c} = T_{I_2}. \qquad \square$$

Remark 5.27. Now let $\{f_I\}_{I \in \mathcal{J}}$ be a locally finite smooth partition of unity of $(\mathbb{R}^d)^n \backslash \Delta_n$ subordinate to $\{\mathcal{C}_I\}_{I \in \mathcal{J}}$. Also recall that by remarks 5.20 and 5.22 we have defined T_I as an operator-valued distribution on $\mathcal{C}_I$ with domain $D_{\mathscr{S}}$. We can extend it to all of $(\mathbb{R}^d)^n \setminus \Delta_n$ by $f \mapsto f_I T_I(f) = T_I(f_I f)$, where f is a smooth test function with compact support in $(\mathbb{R}^d)^n \setminus \Delta_n$. Since the set $\mathcal{J}$ is finite we can just add all $f_I T_I$ to get

$${}^0T(\mathcal{L}_1(x_1) \ldots \mathcal{L}_n(x_n)) := \sum_{I \in \mathcal{J}} f_I T_I \tag{5.14}$$

which is an operator-valued distribution on $(\mathbb{R}^d)^n \setminus \Delta_n$. Now, the main result of this section is the following (Thm. 4.3 of [BF00]).

Theorem 5.28. *The expression in Eq. (5.14) is a well-defined operator-valued distribution on $(\mathbb{R}^d)^n \backslash \Delta_n$ with domain $D_{\mathscr{S}} \subset \mathcal{H}$. It does not depend on the choice of the partition $\{f_I\}_{I \in \mathcal{J}}$ and satisfies Properties 1-5 (on $(\mathbb{R}^d)^n \backslash \Delta_n$).*

Proof. We have already seen that ${}^0T$ is an operator-valued distribution. Hence let us show the independence on the choice of the partition of unity. To this end we choose a second partition $\{f'_I\}_{I \in \mathcal{J}}$. For $x \in (\mathbb{R}^d)^n \backslash \Delta_n$ let $\mathcal{K} = \{I \in \mathcal{J} \,|\, x \in \mathcal{C}_I\}$. Since K is finite and the $\mathcal{C}_I$ are open, there is an open neighbourhood V of x such that $V \subset \cap_{I \in \mathcal{K}} \mathcal{C}_I$. Similarly $\mathcal{J} \setminus \mathcal{K}$ is finite, too, and we can choose V such that

$$V \cap \operatorname{supp}(f_I) = V \cap \operatorname{supp}(f'_I) = \emptyset. \tag{5.15}$$

This implies

$$\sum_{I \in \mathcal{J}} (f_I - f'_I) T_I|_V = \sum_{I \in \mathcal{K}} (f_I - f'_I) T_I|_V.$$

By Prop. 5.26 the restriction of T_I to V is independent of the choice of $I \in \mathcal{K}$. Due to Eq. (5.15) and by the property of the F_I, F_I' we have $\sum_{I\in\mathcal{K}} f_I = \sum_{I\in\mathcal{K}} f_I' = 1$ on V, and therefore the definition of ${}^0T$ does not depend on the choice of the f_I, as stated. This shows *Property 1*. For later reference let us also note that our arguments imply that the restriction of ${}^0T$ coincides with the corresponding restriction of T_I for any $I \in \mathcal{K}$. In other words ${}^0T|_V = (T_I)|_V$ holds for all $I \in \mathcal{K}$.

To show permutation invariance *(Property 2)* we look at the permuted distribution ${}^0T^\pi(x_1,\dots,x_n) = {}^0T(\mathcal{L}_{\pi(1)}(x_{\pi(1)})\dots\mathcal{L}_{\pi(n)}(x_{\pi(n)}))$. Since $\mathcal{J}$ is invariant under permutations we get the expansion

$$ {}^0T^\pi = \sum_{I\in\mathcal{J}} f_I^\pi T_I^\pi = \sum_{I\in\mathcal{J}} f_{\pi(I)}^\pi T_{\pi(I)}^\pi. $$

But $T_{\pi(I)}^\pi = T_I$ and $\{f_{\pi(I)}^\pi\}_{I\in\mathcal{J}}$ is a partition of unity subordinate to $\{\mathcal{C}_I\}_{I\in\mathcal{J}}$. Hence permutation invariance follows from the independence of ${}^0T$ on the choice of the f_I as just proven.

The next step concerns causality *(Property 3)*. For an arbitrary $x \in (\mathbb{R}^d)^n\backslash\Delta_n$ we reuse the set $\mathcal{K} \subset \mathcal{J}$ and the neighborhood V of x introduced above. Hence $x \in V \subset \cap_{I\in\mathcal{K}}\mathcal{C}_I$. By Eq. (5.15) we have ${}^0T(x) = T_I(x)$ which from Remark 5.25 satisfies causality by definition.

For Property 4 (translation invariance) consider $a \in \mathbb{R}^d$ and the action $(x_1,\dots,x_n) \mapsto (x_1+a,\dots,x_n+a)$ on $(\mathbb{R}^d)^n$, and note that the sets $\mathcal{C}_I$ are invariant under this action. The same is true for the T_I since they are defined *(Def. 5.22)* as the product of two terms which are translation invariant under the induction hypothesis. Furthermore, the translated functions $f_{I,a}$ given by $f_{I,a}(x_1,\dots,x_n) = f(x_1+a,\dots,x_n+a)$ are again a partition of unity. Hence by the independence of ${}^0T$ on the choice of the f_I we get

$$ \begin{aligned} {}^0T(x_1+a,\dots,x_n+a) &= \sum_{I\in\mathcal{J}} f_I(x_1+a,\dots,x_n+a)T_I(x_1+a,\dots,1x_n+a) \\ &= \sum_{I\in\mathcal{J}} f_{I,a}(x_1,\dots,x_n)T_I(x_1,\dots,x_n) = {}^0T(x_1,\dots,x_n) \end{aligned} $$

which shows translation invariance.

Finally, Property 5 follows from Remark 5.25 by a straightforward application of the generalized Wick Theorem. □

5.4 Extension to the Diagonal

The remaining step is to extend ${}^0T$ to the diagonal Δ_n. This task is simplified by two facts.

- The form of ${}^0T$ on $\mathbb{R}^{n\cdot d}\backslash\Delta_n$ by Property 5 is

$$T(\mathcal{L}_1(x_1)\dots\mathcal{L}_n(x_n)) = \sum_{j_1,\dots,j_n} \langle\Omega_\omega, T(\mathcal{L}_1^{(j_1)}(x_1)\dots\mathcal{L}_n^{(j_n)}(x_n))\Omega\omega\rangle \frac{:\Phi_m^{j_1}(x_1)\dots\Phi_m^{j_n}(x_n):}{j_1!\dots j_n!}.$$

  We only have to extend the numerical distribution, i.e. the vacuum expectation value of T as the fraction on the right-hand side is defined on all of $\mathbb{R}^{n\cdot d}$ anyway.

- By translation invariance and since the translation group acts transitively, we only have to extend $\tilde{T}_n(y_1,\dots,y_{n-1})$ to $y_1 = \dots = y_{n-1} = 0$ as $T(\mathcal{L}_1(x_1),\dots,\mathcal{L}_n(x_n)) = T_n(x_1,\dots,x_n)$ and because we have translation invariance

$$\tilde{T}_n(x_1 - x_2,\dots,x_{n-1} - x_n) = T_n(x_1,\dots,x_n),$$

  cf. discussion of Wightman distributions.

Hence the message is that we have to extend a distribution on $\mathbb{R}^d \setminus \{0\}$ to the origin. Again we will closely follow [BF00], while they are following Steinmann [Ste71]. Our first step is to introduce a tool which allows us to measure the strength of a singularity at the origin.

Definition 5.29 (Dilation of function)**.** *Define*

$$\begin{aligned}\Lambda : \mathbb{R}_+ \times \mathscr{D}(\mathbb{R}^d) &\to \mathscr{D}(\mathbb{R}^d)\\ (\lambda,\phi) &\mapsto \phi^\lambda := \lambda^{-d}\phi(\lambda^{-1}\cdot).\end{aligned}$$

and for $t \in \mathscr{D}'(\mathbb{R}^d)$

$$t_\lambda(\phi) =: t(\phi^\lambda).$$

Remark 5.30. Please convince yourself that for $t \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ and the corresponding distribution $t(\phi) = \int t(x)\phi(x)dx$ the operation from Definition 5.29 is given by

$$t_\lambda(\phi) = \int t(\lambda x)\phi(x)\, d^d x, \tag{5.16}$$

where in both equations ϕ is an arbitrary test function $\phi \in \mathscr{D}(\mathbb{R}^d)$.

Definition 5.31. *We say that* $t \in \mathscr{D}'(\mathbb{R}^d)$ *has scaling degree* $\mathrm{sd}(t) = \omega$ *with respect to the origin in* $\mathbb{R}^d$ *if* ω *is the infimum over all* $\omega' \in \mathbb{R}$ *for which*

$$\lim_{\lambda\downarrow 0} \lambda^{\omega'} t_\lambda(f) = 0$$

holds for all $f \in \mathscr{D}(\mathbb{R}^d)$.

Remark 5.32. Strictly speaking we have defined $\mathrm{sd}(t)$ only for distributions which are defined on all of $\mathbb{R}^d$. However, since the test function space $\mathscr{D}(\mathbb{R}^d \setminus \{0\})$ is invariant

under the dilations from Def. 5.29, we can easily extend the definition to elements from $\mathscr{D}'(\mathbb{R}^d \setminus \{0\})$, i.e. the distributions we want to extend to the origin.

Example 5.33. To get a better understanding of the concept just introduced, ket us have a short look at some examples.

1. (Regular distribution). Consider $t \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ which is continuous in the origin, and the corresponding regular distribution. In other words for any test function $f \in \mathscr{D}(\mathbb{R}^d)$ the quantity $t_\lambda(f)$ is given by Eq. (5.16). Continuity of t at 0 implies that $\lim_{\lambda \downarrow 0} t(\lambda x) = t(0)$ for all $x \in \mathbb{R}^d$. Since f is compactly supported, dominated convergence implies that
$$\lim_{\lambda \downarrow 0} t_\lambda(f) = t(0) \int_{\mathbb{R}^d} f(x) dx.$$
Hence $\lim_{\lambda \downarrow 0} \lambda^{\omega'} t_\lambda(f) = 0$ holds for all $\omega' > 0$ such that the scaling degree becomes $\mathrm{sd}(t) \leq 0$.
2. (Dirac measure). Recall that $\delta \in \mathscr{S}'(\mathbb{R}^d) \subset \mathscr{D}'(\mathbb{R}^d)$ is given by $\delta(f) = f(0)$ for any $f \in \mathscr{S}(\mathbb{R}^d)$. Hence, with the dilation f^λ we get according to Def. 5.29 $\delta(f^\lambda) = \lambda^{-d} f(0)$. By choosing $f(0) > 0$ we see that
$$\lim_{\lambda \downarrow 0} \lambda^{\omega'} \delta_\lambda(f) = \lambda^{\omega' - d} f(0) = 0 \tag{5.17}$$
holds iff $\omega' > d$, which implies $\mathrm{sd}(\delta) = d$.
3. (Derivative of Dirac measure). We can extend the previous example with a polynomial P on $\mathbb{R}^d$ of degree p. It defines the partial differential operator
$$P(\partial) f(x) = \sum_{|\alpha| \leq p} c_\alpha D_\alpha f$$
with $c_\alpha \in \mathbb{C}$. Applying $P(\partial)$ to δ leads to (cf. Example 1.8)
$$P(\partial)\delta(f) = \sum_{|\alpha| \leq p} c_\alpha (-1)^{|\alpha|} D_\alpha f(0) \quad f \in \mathscr{S}(\mathbb{R}^d). \tag{5.18}$$
We can choose f such that only the derivatives of order p are non-zero at $x = 0$, i.e.
$$\begin{aligned} P(\partial)\delta(f^\lambda) &= (-1)^p \sum_{|\alpha| = p} c_\alpha (-1)^{|\alpha|} D_\alpha f_\lambda(0) \\ &= (-1)^p \lambda^{-(p+d)} \sum_{|\alpha| = p} c_\alpha D_\alpha f(0) = \kappa \lambda^{-(p+d)} \end{aligned}$$
with some $\kappa \neq 0$. Hence as in Eq. (5.17) we find $\mathrm{sd}\big(P(\partial)\delta\big) = d + p$.
4. (Feynman propagator). The Feynmann propagator is (roughly speaking) the time-

ordered two point function. Hence for the free scalar field we get

$$E_F(x_1 - x_2) = \langle \Omega_0, T(\Phi_m(x_1)\Phi_m(x_2))\Omega_0 \rangle.$$

It can be written as

$$E_F(x_1 - x_2) = (2\pi)^{-d} \int \frac{e^{ip\cdot(x_1-x_2)}}{p^2 - m^2 + i\varepsilon}\, d^d p.$$

A short calculation shows that $\mathrm{sd}(E_F) = d - 2$ holds.

5. (Homogeneous distributions). If $t \in \mathscr{D}'(\mathbb{R}^d)$ is homogeneous of order α at the origin, i.e. $t_\lambda = \lambda^\alpha t$, then $\mathrm{sd}(t) = -\alpha$. In other words the scaling measures up to a certain degree the homogeneity of a distribution.
6. (Infinite degree). The smooth function $x \to \exp(1/x)$, $x \in \mathbb{R}_+$ is not defined at the origin and its scaling degree with respect to the origin is clearly infinite.

The following lemma (cf. Lemma 5.1. of [BF00]) summarizes some properties of the scaling degree. They are needed for the theorems presented below.

Lemma 5.34. *The scaling degree obeys the following properties.*

(a) Let $t \in \mathscr{D}'(\mathbb{R}^d)$ have $\mathrm{sd}(t) = \omega$ at 0, then
   1. *Let $\alpha \in \mathbb{N}^n$ be any multiindex, then $\mathrm{sd}(\partial^\alpha t) \leq \omega + |\alpha|$.*
   2. *Let $\alpha \in \mathbb{N}^n$ be any multiindex, then $\mathrm{sd}(x^\alpha t) \leq \omega - |\alpha|$.*
   3. *Let $f \in C^\infty(\mathbb{R}^d)$, then $\mathrm{sd}(ft) \leq \mathrm{sd}(t)$.*

(b) For $t_i \in \mathscr{D}'(\mathbb{R}^{d_i})$, $i = 1, 2$ we have
   1. *$\mathrm{sd}(t_1 \otimes t_2) = \mathrm{sd}(t_1) + \mathrm{sd}(t_2)$,*
   2. *and if $d_1 = d_2 = d$ we have $\mathrm{sd}(t_1 + t_2) \leq \max\big(\mathrm{sd}(t_1), \mathrm{sd}(t_2)\big)$.*

Proof. All statements are straightforward, except the third case in (a). Here, we refer the reader to the proof of Lemma 5.1. in [BF00]. □

Remark 5.35. Now we want to extend distributions using the scaling degree. There are three possible cases. When the scaling degree is $+\infty$, then no extension to a distribution on $\mathbb{R}$ exists. When the scaling degree ω is finite, but $\omega \geq d$ then a finite-dimensional set of extensions exists. If $\omega < d$ holds, there is a unique extension. We first study this case.

Remark 5.36. There are two technical details about distributions which are important in the proof of the next theorem. Since we do not use them otherwise, we add this remark here rather in Sect. 1.1.

1. The support of δ contains just the origin and this does not change if we look at $P(\partial)\delta$ with a differential operator as in Eq. (5.18). The more non-trivial fact is that these are the only distributions $T \in \mathscr{D}(\mathbb{R}^d)$ with $\mathrm{supp}(T) = \{0\}$; cf. Thm V.11 of [RS80].

2. The space $\mathscr{D}'(\mathbb{R}^d)$ is equipped with the weak-$*$-topology. Hence a sequence $T_n \in \mathscr{D}'(\mathbb{R}^d)$, $n \in \mathbb{N}$ converges to a distribution $T \in \mathscr{D}'(\mathbb{R}^d)$ iff $\lim_{n\to\infty} T_n(f) = T(f)$ holds for all $f \in \mathscr{D}(\mathbb{R}^d)$. If on the other hand we only know that all sequences $T_n(f)$, $n \in \mathbb{N}$ with arbitrary test functions f are Cauchy sequences we can define a linear functional T by $T(f) = \lim_n T_n(f)$. This T is not just linear but even continuous, i.e. $T \in \mathscr{D}'(\mathbb{R}^d)$. This fact is called sequential completeness of $\mathscr{D}'(\mathbb{R}^d)$. To prove it requires knowledge from the topology of $\mathscr{D}(\mathbb{R}^d)$. A possible strategy is to use Prop. 1.16 together with the uniform boundedness principle; cf. Thm 1.45 of [HS09].

Theorem 5.37 (Thm. 5.2 of [BF00]). *Let $t_0 \in \mathscr{D}'(\mathbb{R}^d\backslash\{0\})$ have scaling degree $\omega < d$ with respect to the origin. There exists unique $t \in \mathscr{D}'(\mathbb{R}^d)$ with scaling degree ω such that $t(\phi) = t_0(\phi)$ for all $\phi \in \mathscr{D}(\mathbb{R}^d\backslash\{0\})$.*

Proof. Uniqueness: Assume there is a second $\tilde{t}$ with $(t - \tilde{t})|_{\mathbb{R}^d\backslash\{0\}} = 0$. This implies $\operatorname{supp}(t - \tilde{t}) = \{0\}$ so by Rem. 5.36 $t - \tilde{t}$ is given by $P(\partial)\delta$ with is a polynomial P of degree p and the delta distribution δ. By the discussion in Ex. 5.33 this distribution has scaling degree $d + p$ (cf. also Lem. 5.34). Hence if $P \neq 0$ we get $\operatorname{sd}(t - \tilde{t}) = d + p \geq d$. Since by assumption t and $\tilde{t}$ have a scaling degree smaller than d this contradicts the last statement in Lem. 5.34. Hence $P = 0$ and $t = \tilde{t}$.

Existence: Let us now consider a smooth function θ of compact support such that $\theta = 1$ in a neighbourhood of the origin. Set $\theta_\lambda(x) := \theta(\lambda x)$, $\lambda \in \mathbb{R}$ and

$$t^{(n)} := (1 - \theta_{2^n})t_0 \qquad n \in \mathbb{N}$$

where now $t^{(n)}$ is a sequence of distributions defined on the whole $\mathbb{R}^d$. We wish to show that the sequence converges in the weak-$*$-topology of $\mathscr{D}'(\mathbb{R}^d)$. According to Rem. 5.36 it is suffcient to show that for all $f \in \mathscr{D}(\mathbb{R}^n)$ the numbers $t^{(n)}(f)$, $n \in \mathbb{N}$ form a Cauchy sequence. Hence we look at

$$(t^{(n+1)} - t^{(n)})(f) = (ft_0)(\theta_{2^n} - \theta_{2^{n+1}}) = 2^{-nd}(ft_0)_{2^{-n}}(\theta - \theta_2).$$

Here we have used two facts. Firstly the product of a distribution T with a function ϑ is given by $(\vartheta T)(f) = T(\vartheta f)$; cf. Ex. 1.8. If ϑ is a valid test function we can reverse the roles of f and ϑ to get $(\vartheta T)(f) = (fT)(\vartheta)$. Secondly we have used the definition of the scaled distribution T_λ in Def. 5.29 to extract a factor $(2^{-n})^d$. Rewriting this equation with $\omega' \in \mathbb{R}$ we get

$$(t^{(n+1)} - t^{(n)})(f) = 2^{-n(d-\omega')}\big(2^{-n\omega'}(ft_0)_{2^{-n}}(\theta - \theta_2)\big).$$

According to Lemma 5.34 (a.3) we have $\operatorname{sd}(ft_0) \leq \omega$. Hence if $\omega' > \omega$ the definition of the scaling degree implies that the sequence $\big|2^{-n\omega'}(t_0)_{2^{-n}}(\theta - \theta_2)\big|$ converges to zero and is in particular bounded by a constant C. If we choose ω' smaller than d (which is possible due to $\omega < d$ by assumption), we get the bound

$$\big|t^{(n+1)} - t^{(n)}(f)\big| \leq C2^{-n(d-\omega')}$$

with an absolutely summable sequence $2^{-n(d-\omega')}$. This implies that for any $\epsilon > 0$ we can find an $N_\epsilon \in \mathbb{N}$ such that

$$\left|t^{(n)}(f) - t^{(m)}(f)\right| \leq \left|t^{(n)}(f) - t^{(n+1)}(f)\right| + \cdots + \left|t^{(m-1)}(f) - t^{(m)}(f)\right| \leq \sum_{j>N_\epsilon} 2^{-j(d-\omega')} < \epsilon$$

holds for $n, m > N_\epsilon$ and $m > n$. Hence $t^{(n)}(f)$ is for all $f \in \mathscr{D}(\mathbb{R}^d)$ a Cauchy sequence, such that the limits $\lim_{n\to\infty} t^{(n)}(f) =: t(f)$ exists and define a distribution $t \in \mathscr{D}'(\mathbb{R}^d)$ which is defined on all of $\mathbb{R}^d$.

To see that $t(f) = t_0(f)$ for all $f \in \mathscr{D}(\mathbb{R}^d \setminus \{0\})$ holds, note that there is an open neighborhood $V \subset \mathbb{R}^d$ of the origin with $\operatorname{supp} f \cap V = \emptyset$. Since θ is compactly supported there is an $N \in \mathbb{N}$ with $\operatorname{supp}\left(\theta_{2^n}\right) \subset V$ for all $n > N$. Hence $(1 - \theta_{2^n})f = f$ for all such n and we get $t(f) = \lim_{n\to\infty} t^{(n)}(f) = t_0(f)$ by the definition of $t^{(n)}$.

This finishes the construction of the extension t of t_0. It remains to proof that the scaling degree of t is ω. For this we refer to [BF00]. □

Remark 5.38. We now deal with the extension procedure in case a distribution has a finite scaling degree $\omega \geq d$. This extension procedure corresponds to renormalization in other schemes. To adhere more to the standard notation we introduce the degree of singularity $\rho := \omega - d$. This is the analog of the degree of divergence of a Feynman diagram.

Remark 5.39. Let $\mathscr{D}_\rho(\mathbb{R}^d)$ be the set of all smooth functions of compact support which vanish of order ρ at the origin. The space of distributions T satisfying $T(f) = 0$ for all $f \in \mathscr{D}_\rho(\mathbb{R}^d)$ – i.e. the orthocomplement of $\mathscr{D}_\rho(\mathbb{R}^d)$ in $\mathscr{D}'(\mathbb{R}^d)$ – is spanned by derivatives of the delta distributions (please check yourself). Hence a family of test functions $\mathfrak{w}_\alpha \in \mathscr{D}(\mathbb{R}^d)$ with $D^\alpha \mathfrak{w}_\beta = m\delta^\alpha_\beta$ for all multiindices α, β satisfying $|\alpha|, |\beta| \leq \rho$ spans an algebraic complement of $\mathscr{D}_\rho(\mathbb{R}^d)$ in $\mathscr{D}(\mathbb{R}^d)$. This complement is finite-dimensional, since the orthocomplement is finite-dimensional, too. Hence we get a projection W from $\mathscr{D}(\mathbb{R}^d)$ onto $\mathscr{D}_\rho(\mathbb{R}^d)$, by

$$Wf = f - \sum_{|\alpha|\leq\rho} \mathfrak{w}_\alpha \partial^\alpha f(0).$$

Now the idea is to extend t_0 to a functional on $\mathscr{D}_\rho(\mathbb{R}^d)$ with ρ sufficiently large and then to use a projection W. This procedure is implemented in the next theorem.

Theorem 5.40 (Thm. 5.3 of [BF00])**.** *Let $t_0 \in \mathscr{D}'(\mathbb{R}^d \setminus \{0\})$ have a finite scaling degree $\omega \geq d$. Then there exist extensions $t \in \mathscr{D}'(\mathbb{R}^d)$ of t_0 with the same scaling degree. They are uniquely determined by their values on the test functions $\mathfrak{w}_\alpha$ chosen in the previous remark.*

Proof. Since the $\mathfrak{w}_\alpha$ span an algebraic complement of $\mathscr{D}_\rho(\mathbb{R}^d)$, any $f \in \mathscr{D}(\mathbb{R}^d)$ can be uniquely decomposed as $f = f_1 + f_2$ where $f_1 = \sum_{|\alpha|\leq\rho} \mathfrak{w}_\alpha \partial^\alpha f(0)$ and $f_2 \in \mathscr{D}_\rho(\mathbb{R}^d)$ has

the form

$$f_2(x) = \sum_{|\alpha|=\rho+1} x^\alpha g_\alpha(x)$$

where $g_\alpha \in \mathscr{D}(\mathbb{R}^d)$. This can be checked with the Taylor expansion of f_2 around 0 (it does not converge since f is compactly supported, hence we have to use a remainder term). By assumption all terms up to order ρ vanish and therefore we can extract the monomials x^α with $\alpha = \rho + 1$ leaving smooth quotients g_α. Hence a general element T of $\mathscr{D}'(\mathbb{R}^d)$ can be written as

$$t(f) = \sum_{|\alpha|=\rho+1} \big(x^\alpha t_0\big)(g_\alpha) + t(f_1),$$

where the last term is uniquely determined by the values $t(\mathfrak{w}_\alpha) = c_\alpha$. Hence we get

$$t(f) = \sum_{|\alpha|=\rho+1} \big(x^\alpha t_0\big)(g_\alpha) + \sum_{|\alpha|\leq\rho} c_\alpha \partial^\alpha f(0). \tag{5.19}$$

Please check that we really get $t(\mathfrak{w}_\alpha) = c_\alpha$ with the decomposition of f and the expression in Eq. (5.19). To proceed, note that all $f \in \mathscr{D}(\mathbb{R}^d)$ with $\operatorname{supp} f \subset \mathbb{R}^d \setminus \{0\}$ vanish on a neighborhood of the origin. Hence $f \in \mathscr{D}_\rho(\mathbb{R}^d)$ for all ρ. This implies $f = f_2$ for such f and the corresponding g_α still have support in $\mathbb{R}^d \setminus \{0\}$. Hence, if t is an extension of t_0 it has to satisfy

$$t(f) = \sum_{|\alpha|=\rho+1} \big(x^\alpha t\big)(g_\alpha) = \sum_{|\alpha|=\rho+1} \big(x^\alpha t_0\big)(g_\alpha).$$

Now note that by Lemma 5.34, $x^\alpha t_0$ has scaling degree smaller than or equal to $\operatorname{sd}(t_0) - \rho - 1$. We can choose ρ such that this quantity is strictly smaller than d and then all the distributions $x^\alpha t_0$ have a unique extensions $\tilde{t}_\alpha$ to $\mathscr{D}(\mathbb{R}^d)$; cf. Thm. 5.37. This reasoning shows that all distributions of the form

$$t(f) = \sum_{|\alpha|=\rho+1} \tilde{t}_\alpha(g_\alpha) + \sum_{|\alpha|\leq\rho} c_\alpha \partial^\alpha f(0),$$

with $\tilde{t}_\alpha$ just defined and arbitrary $c_\alpha \in \mathbb{C}$ are extensions of t_0 and that any extension has to be of this form. Each of those t is uniquely determined by the c_α which we have already seen to be equal to $t(\mathfrak{w}_\alpha)$. This proves the existence and uniqueness statement. For the remaining part of the proof, i.e. to show that $\operatorname{sd}(t) = \operatorname{sd}(t_0)$ holds, we refer again to [BF00]. □

Remark 5.41. Let us come back to the results of the previous section. There we have constructed the time ordered products $T(\mathcal{L}_1(x_1) \dots \mathcal{L}_n(x_n))$ as operator valued distribu-

tions on the set $(\mathbb{R}^d)^n \setminus \Delta_n$. They can be expanded as

$$T(\mathcal{L}_1(x_1)\dots\mathcal{L}_n(x_n)) = \sum_{j_1,\dots,j_n} \langle\Omega_\omega, T(\mathcal{L}_1^{(j_1)}(x_1)\dots\mathcal{L}_n^{(j_n)}(x_n))\Omega_\omega\rangle \frac{:\Phi_m^{j_1}(x_1)\dots\Phi_m^{j_n}(x_n):}{j_1!\dots j_n!} \tag{5.20}$$

where the Wick monomials $:\Phi_m^{j_1}(x_1)\dots\Phi_m^{j_n}(x_n):$ are well defined operator valued distributions on all of $(\mathbb{R}^d)^n$. Only the numerical diustributions $\langle\Omega_\omega, T(\mathcal{L}_1^{(j_1)}(x_1)\dots\mathcal{L}_n^{(j_n)}(x_n))\Omega_\omega\rangle$ need further treatment. But we know that they are translation invariant such that we can introduce distributions $t_0^{j_1,\dots,j_n} \in \mathscr{D}'\big((\mathbb{R}^d)^{n-1}\setminus\{0\}\big)$ satisfying

$$\langle\Omega_\omega, T(\mathcal{L}_1^{(j_1)}(x_1)\dots\mathcal{L}_n^{(j_n)}(x_n))\Omega_\omega\rangle = t_0^{j_1,\dots,j_n}(x_1-x_2,\dots,x_{n-1}-x_n). \tag{5.21}$$

In other words we are factoring out the orbits of the translation group. Therefore the diagonal Δ_n reduces to the origin and the $t_0^{j_1,\dots,j_n}$ are defined on $(\mathbb{R}^d)^{n-1}\setminus\{0\}$. Now we can apply the extension procedure just studied to get distributions $t^{j_1,\dots,j_n}$ defined on all of $(\mathbb{R}^d)^{n-1}$ – including the origin, and via Eqs. (5.20) and (5.21) we can extend the time ordered products $T(\mathcal{L}_1(x_1)\dots\mathcal{L}_n(x_n))$ to the diagonal Δ_n.

This finishes our construction and we end up with the perturbative construction of the S-matrix as (cf. Eq. (5.7) in Sect. 5.2)

$$S(g) = \mathbb{1} + \sum_{n=1}^{\infty} \frac{(-i)^n}{n!} \int_{\mathbb{R}^d}\dots\int_{\mathbb{R}^d} g(y_1)\dots g(y_n) T\{:\Phi_m(y_1)^4:\dots:\Phi_m(y_n)^4:\}\, dy_n\dots dy_1.$$

This is now a formal power series in the coupling g with coefficients $T\{:\Phi_m(y_1)^4:\dots:\Phi_m(y_n)^4:\}$ which are well defined operator valued distributions on $(\mathbb{R}^d)^n$. Hence, for a given g and state vectors $\psi,\phi\in\mathcal{H}$ in the Hilbert space $\mathcal{H} = \mathcal{F}_+\big(\mathrm{L}^2(H_m)\big)$ of the free field we can calculate the scattering amplitude $\langle\phi, S(g)\psi\rangle$ perturbatively term by term. In general the series we get that way does not converge, but we still can get reliable predictions by stopping the expansion after finitely many orders (cf. the discussion in Sects. 5.1 and 5.2). Since ψ, ϕ are states of the free field we automatically have an interpretation in terms of particles (cf. the corresponding discussion in Sect. 4.4).

At that point an additional problem arises. The extension of Thm. 5.40 is not unique but depends on the parameters $c_\alpha = t(\mathfrak{w}_\alpha)$ from Eq. (5.19) we have to choose. This introduces more free parameters into our model. To many parameters are bad, since if we have too much of them we can fit basically everything (like the famous "elephant"). Therefore theories are divided into two categories depending on the number of free parameters renormalization introduces. If it remains finite the theory is called renormalizable, otherwise non-reonormalizable. A detailed analysis of the scaling degree leads to conditions for renormalizability cf. [EG73]. The Φ^4 model we have looked at is renomalizable in four dimensions.

The next step would be to present some calculations, however, we skip this part (at least in the present version of this document) and refer the reader to the thesis of Pinter [Pin00].

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