Information-Disturbance Tradeoff

Anna-Lena K. Hashagen & Michael M. Wolf

Fakultät Mathematik Technische Universität München hashagen@ma.tum.de

The information-disturbance tradeoff describes the inevitable disturbance caused to a quantum system when information is gained through a measurement. It can be cast as a special case of a more general problem, in which we consider the problem of approximate simultaneous realisations of two arbitrary (and not necessary compatible) quantum operations. In general this leads to a trade-off where increasing quality of one operation necessarily decreases the quality of the other. We prove that a tight trade-off bound on the quality of the two approximate simultaneous realisations (when measured in terms of the cb norm) can be computed in terms of a semi-definite program (SDP). For the special case of the information-disturbance tradeoff, the resulting SDP allows us to obtain analytic results for binary von Neumann measurements.

Setting and distance measures

Consider systems on a finite dimensional Hilbert space $\mathcal{H} = \mathbb{C}^d$.

Quantum state Every quantum state is described by a density matrix $\rho \in \mathcal{M}_d$ with $\mathrm{Tr}[\rho] = 1$ and $\rho \geq 0$.

Quantum channel A transformation of a quantum state is described by a quantum channel, which is a linear completely positive trace preserving map

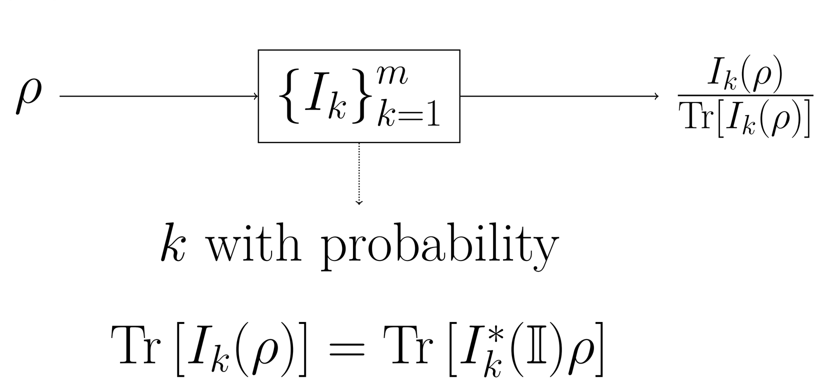
$$T : \mathcal{M}_d \to \mathcal{M}_{d'}.$$

POVM A discrete observable $\mathcal{P}$ is described by a positive-operator valued measure (POVM), characterised by positive operators

$$\mathcal{P} := \{P_k \in \mathcal{M}_d\}_{k=1}^m, \text{ satisfying } 0 \leq P_k \leq \mathbb{I} \ \forall k \text{ and } \sum_{k=1}^m P_k = \mathbb{I}.$$

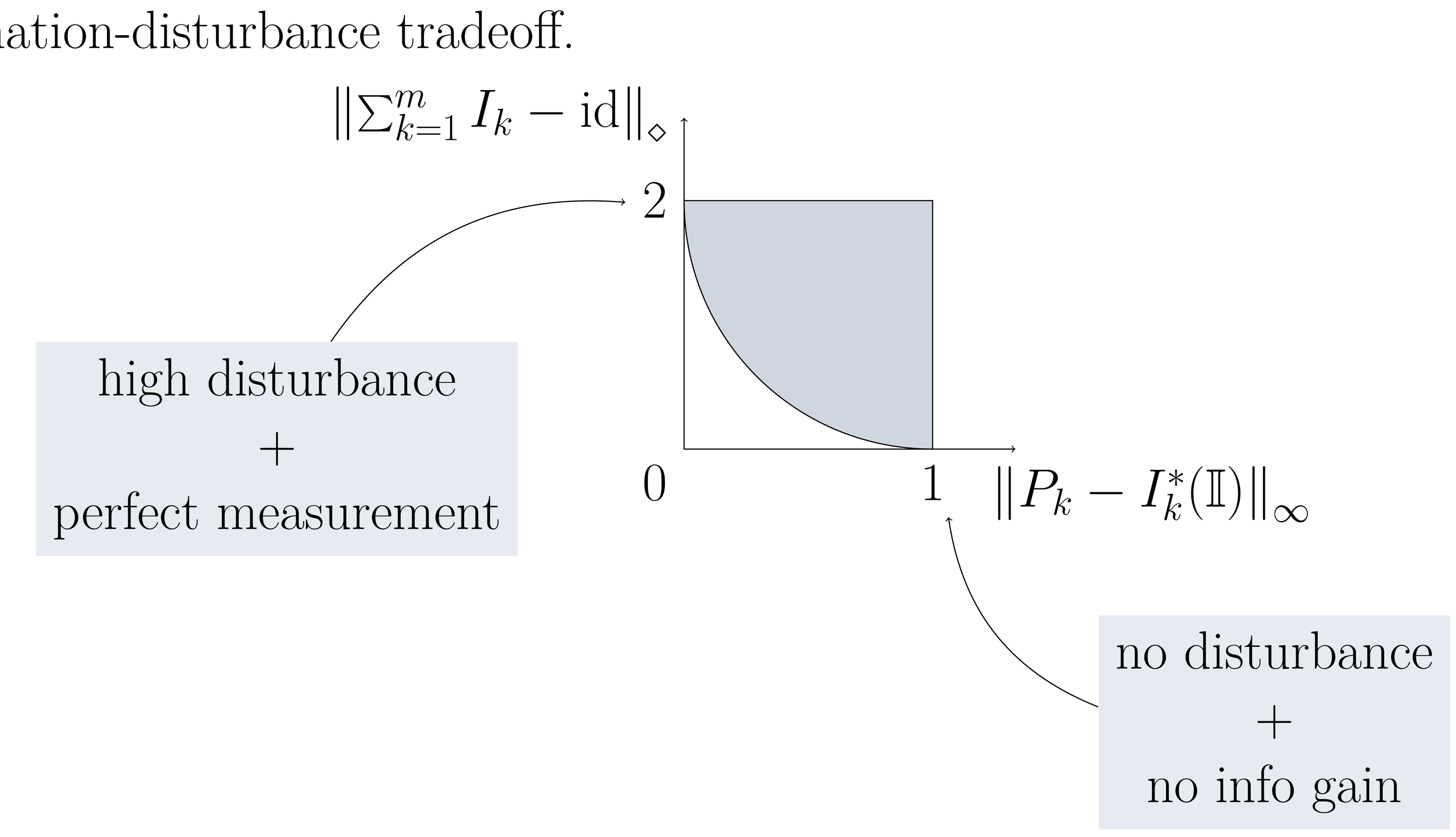
Instrument A general measurement scheme is represented by an instrument $\mathcal{I}$, which is characterised by completely positive linear maps

$$\mathcal{I} := \{I_k : \mathcal{M}_d \to \mathcal{M}_{d'}\}_{k=1}^m, \text{ satisfying } \sum_{k=1}^m I_k^*(\mathbb{I}) = \mathbb{I}.$$



An instrument therefore comprises a quantum channel and a POVM. If we ignore the measurement outcome, the instrument $\mathcal{I}$ acts as a quantum channel, $\sum_k I_k$. If we ignore the quantum state of the output, the instrument $\mathcal{I}$ acts as a POVM, $\{I_k^*(\mathbb{I})\}$.

The goal of this project is to completely characterise the set of all achievable information disturbance combinations and thus determine its boundary, the information-disturbance tradeoff.



Fix a POVM $\mathcal{P} := \{P_k\}_{k=1}^m$ and consider an instrument $\mathcal{I} := \{I_k\}_{k=1}^m$.

❶ The distance measure quantifying the information gained from the quantum system when performing the measurement $\{I_k^*(\mathbb{I})\}$, in comparison to the measurement $\{P_k\}$ is

$$d_1(P_k, I_k^*(\mathbb{I})) = \sup_\rho |\mathrm{Tr}[\rho P_k] - \mathrm{Tr}[I_k^*(\mathbb{I})\rho]| = \|P_k - I_k^*(\mathbb{I})\|_\infty \quad \forall k.$$

This is the worst case over all quantum states ρ of the difference of the probabilities for a specific measurement outcome k to occur regarding the measurements $\{P_k\}$ and $\{I_k^*(\mathbb{I})\}$.

❷ The distance measure quantifying the disturbance caused to the quantum system $\sum_k I_k$ in comparison to the identity channel id is

$$d_2\left(\sum_{k=1}^m I_k, \mathrm{id}\right) = \sup_\rho \left\|\sum_{k=1}^m I_k \otimes \mathrm{id}(\rho) - \mathrm{id}(\rho)\right\|_1 = \left\|\sum_{k=1}^m I_k - \mathrm{id}\right\|_\diamond.$$

We compare the pre-measurement quantum state to the post-measurement quantum state.

Optimisation problem

Task (⋆)

Compute for a given POVM $\mathcal{P}$ and $\lambda \in [0,1]$:

$$\begin{aligned} &\min \left\|\mathrm{id} - \sum_{k=1}^m I_k\right\|_\diamond \\ &\text{s.t. } \|I_k^*(\mathbb{I}) - P_k\|_\infty \leq \lambda \ \forall k, \\ &I_k \text{ is c.p. and} \\ &\sum_{k=1}^m I_k^*(\mathbb{I}) = \mathbb{I} \ \forall k. \end{aligned}$$

Results

Theorem (⋆)

For a given POVM $\mathcal{P}$ and $\lambda \in [0,1]$, the optimisation specified in Task (⋆) can be formulated as an SDP (ϕ, C, D), where $\phi : \mathcal{M}_d \to \mathcal{M}_{d'}$ is a hermiticity preserving map, $C = C^* \in \mathcal{M}_d$ and $D = D^* \in \mathcal{M}_{d'}$, with the primal and the dual SDP problem given as follows

$$\begin{aligned} &\max \mathrm{Tr}[CX] \\ &\text{s.t. } \phi(X) = D \\ &X \geq 0 \end{aligned} \quad = \quad \begin{aligned} &\min \mathrm{Tr}[DY] \\ &\text{s.t. } \phi^*(Y) \geq C \\ &Y = Y^* \end{aligned}$$

Slater-type strong duality holds

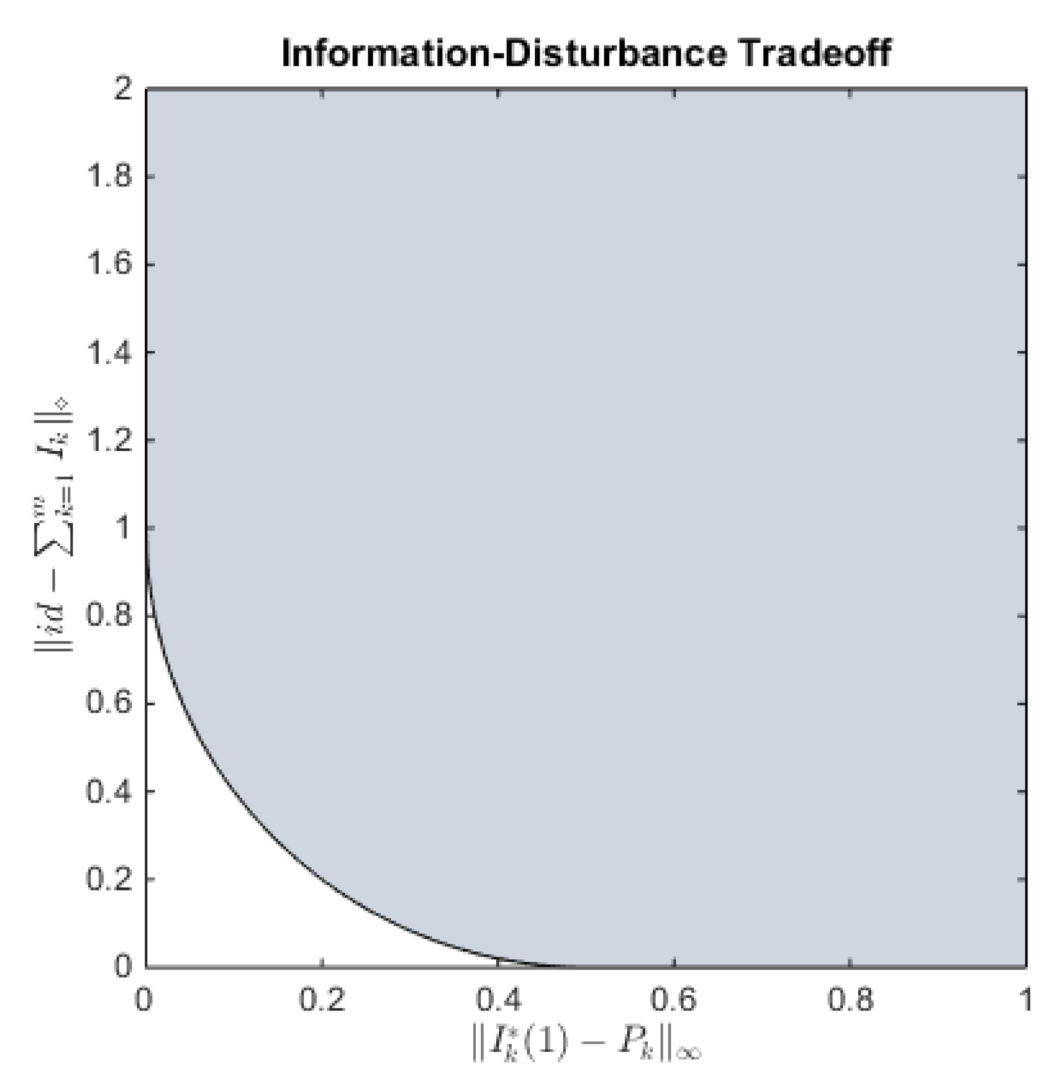


Figure 1: The set of achievable information disturbance combinations for a fixed binary von Neumann measurement $\mathcal{P} = \{P, \mathbb{I} - P\}$. The information-disturbance tradeoff is given by an ellipse

$$d_2\left(\sum_{k=1}^m I_k, \mathrm{id}\right) \geq 1 - 2\sqrt{d_1(P_k, I_k^*(\mathbb{I})) - (d_1(P_k, I_k^*(\mathbb{I})))^2}.$$

Literature

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