How long does it take to obtain a physical density matrix?

Lukas Knips[1,2], Christian Schwemmer[1,2], Nico Klein[1,2], Jonas Reuter[3], Géza Tóth[4,5,6], and Harald Weinfurter[1,2]

[1] Fakultät für Physik, Ludwig-Maximilians-Universität, München, Germany, [2] Max-Planck-Institut für Quantenoptik, Garching, Germany, [3] Bethe Center for Theoretical Physics, Universität Bonn, Germany, [4] Department of Theoretical Physics, The University of the Basque Country, Bilbao, Spain, [5] IKERBASQUE, Basque Foundation for Science, Bilbao, Spain, [6] Wigner Research Centre for Physics, Hungarian Academy of Sciences, Budapest, Hungary

LUDWIG-MAXIMILIANS UNIVERSITÄT MÜNCHEN

experimental quantum physics

http://xqp.physik.uni-muenchen.de

Abstract

Quantum state tomography became the standard tool for fully determining unknown quantum states. However, in experiments, one often obtains density matrices with at least one negative eigenvalue. Since the eigenvalues of a density operator are associated with probabilities, this poses a serious conceptual problem. In principle, numerical procedures like maximum likelihood estimation (MLE) or least squares (LS) fitting allow to overcome this problem, yet, only at the prize of distorted results due to biased state estimation [1,2].

The question arises how unphysical solutions can be avoided. Contrary to frequent folklore, bad experiments and misalignment are surely not the only reasons. Here, we demonstrate how statistical noise alone almost unavoidably causes unphysical estimates. It is shown that for the, e.g., overcomplete Pauli tomography scheme typical Poissonian or multinomial measurement statistics, the distribution of eigenvalues for large number of qubits can be described by the Wigner semicircle distribution [3] where the radius depends on the total number of measurements [4].

Already for small system size (a small numbers of qubits) this semicircle distribution can be used as an adequate approximation. We can now specify how likely an unphysical solution is or alternatively can give a minimum number of measurements necessary to avoid an unphysical result.

Based on that, knowing the distribution of eigenvalues now enables both a new ansatz to obtain a physical density matrix from an unphysical estimate as well as to analyze possible misalignment or colored noise by hypothesis testing [4].

Wigner semicircle distribution

• Quantum State Estimation [5]

- simulations to avoid experimental artefacts
- overcomplete Pauli scheme: project onto all eigenstates of tensor products of Pauli matrices $\sigma_x, \sigma_y, \sigma_z$
- set of 6^n frequencies (normalized counts): $$f_r^s = c_r^s / N_s$$
- linearly reconstructed density matrix with A_r^s given by reconstruction scheme: $$\hat{\varrho}_{\mathrm{LIN}} = \sum_{r,s} A_r^s f_r^s$$
- parametrize in terms of Pauli matrices and correlations: $$\varrho = \frac{1}{2^n} \sum_{\vec{\mu}} T_{\vec{\mu}} \sigma_{\vec{\mu}}$$

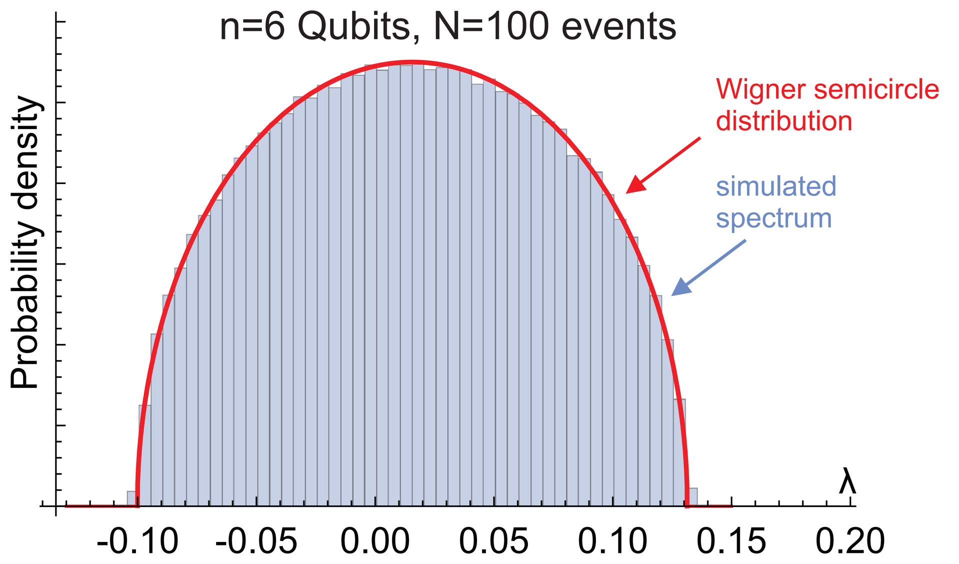
• Maximally Mixed State

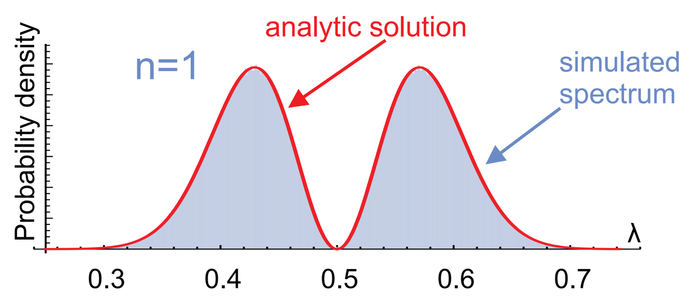
- use maximally mixed state (white noise): $$\varrho_{\mathrm{mm}} = \frac{1}{2^n} \sigma_{0,0,\ldots,0} = \frac{\mathbb{1}}{2^n}$$
- 2^n-fold degenerate eigenvalue $1/2^n$
- suitable to study effect of statistics onto eigenvalues
- (approximative) model given by Wigner semicircle function $$f_{c,R}(x) = \frac{2}{\pi R^2} \sqrt{(x-c)^2 - R^2}$$
- appropriate description solely depends on two parameters:

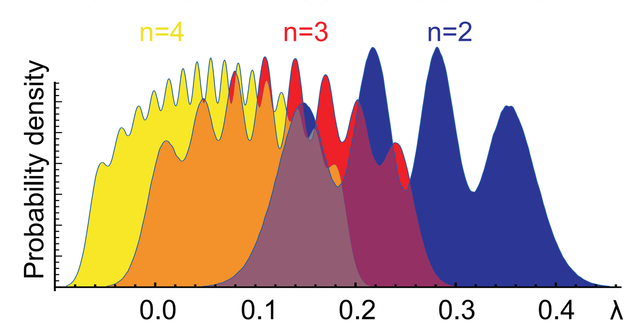
Radius: $$R = 2\sqrt{\frac{10^n - 1}{12^n}} \frac{1}{\sqrt{N}}$$

Center: $$c = \frac{1}{2^n} \left[\sum_{i=1}^{2^n} \lambda_i \right] = \frac{1}{2^n}$$

• Spectral Probability Density







Necessary statistics for $\varrho \geq 0$

• Mixed and pure states

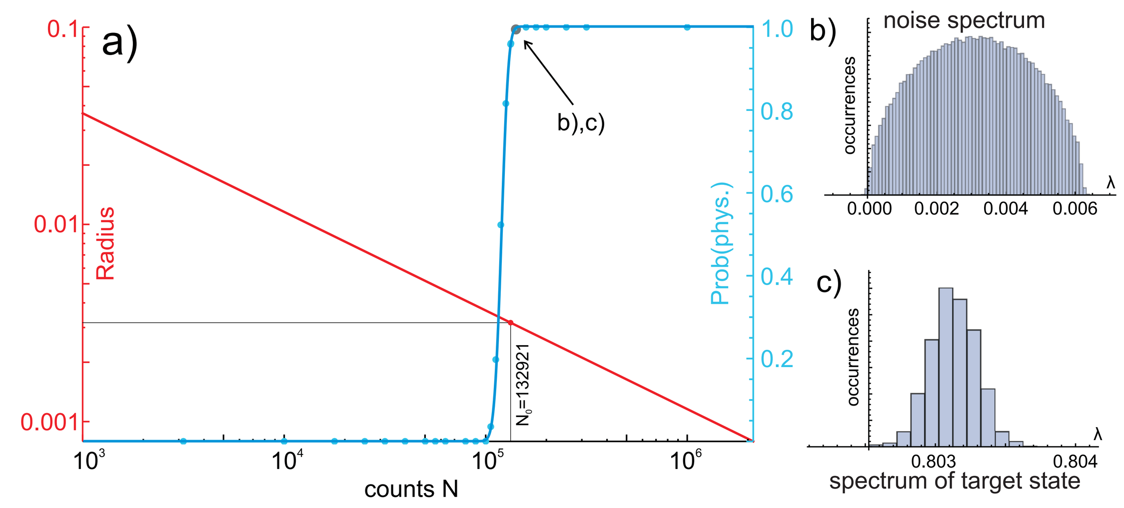
- Noise model: Mixture of pure *target* state $|\psi\rangle$ with *white noise*: $$\varrho_{q,|\psi\rangle} = q|\psi\rangle\langle\psi| + (1-q)\varrho_{\mathrm{mm}}$$
- Center c_q of noise spectrum shifted: $$c_q = \frac{1-q}{2^n - 1}$$
- Radius of distribution of *noise* eigenvalues remains unchanged
- Increasing statistics (N) shrinks radius R
- When is radius R smaller than center c_q? All eigenvalues expected to be positive $$N \geq N_0 = \frac{4(10^n - 1)}{12^n} \left(\frac{2^n - 1}{1-q} \right)^2$$

• Probability for physicality

- Physicality $\varrho \geq 0$: all eigenvalues are non-negative
- Estimate distribution of smallest eigenvalue with Gaussian approximation
- Mean value $$\mu_{\lambda_1} \approx c - R + 2\sigma_{\lambda_1} = c - 2\left(\frac{5}{6}\right)^{n/2} \sqrt{\frac{1}{N}} + 2\sigma_{\lambda_1}$$
- Standard deviation $$\sigma_{\lambda_1} \approx \left(\frac{9}{14}\right)^{n+2} R = \left(\frac{9}{14}\right)^{n+2} 2 \left(\frac{5}{6}\right)^{n/2} \sqrt{\frac{1}{N}}$$
- Probability for smallest eigenvalue being positive $$p(\lambda_1 > 0) = 1 - \int_{-\infty}^{0} \frac{1}{\sqrt{2\pi}\sigma_{\lambda_1}} \exp\left(-\frac{(\lambda - \mu_{\lambda_1})^2}{2\sigma_{\lambda_1}^2}\right) \mathrm{d}\lambda$$ $$= 1 - \frac{1}{2} \operatorname{erfc}\left(\frac{\mu_{\lambda_1}}{\sqrt{2}\sigma_{\lambda_1}}\right)$$

• Example (six qubit state)

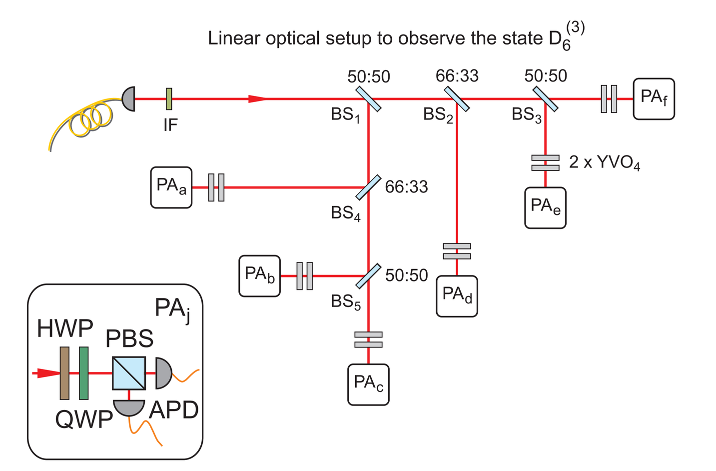
- noisy n=6 qubits GHZ state
- q=0.8 amplitude for GHZ state
- model: N_0=132921 events per setting
- simulation for N_0: 95.9% physical states



Experimental application

• Six-Qubit Tomography [6]

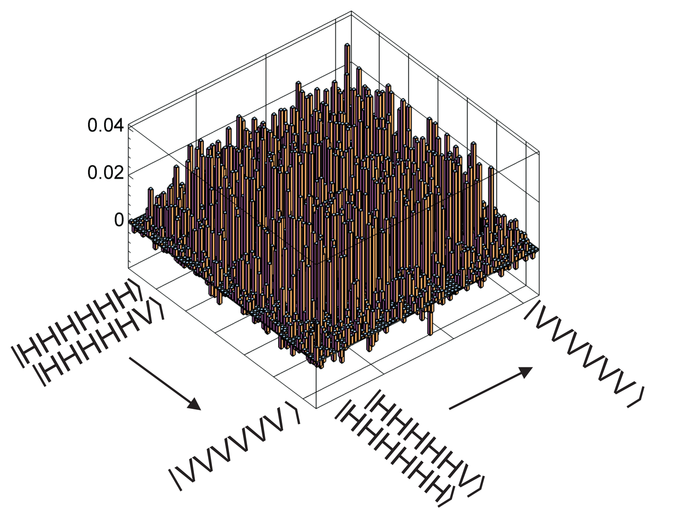
$$|D_6^{(3)}\rangle = \frac{1}{\sqrt{20}} [|000111\rangle + |001011\rangle + \cdots + |111000\rangle]$$



- Spectrum of ideal state: {0,0,...,0,1}
- Spectrum of measured state: {-0.064, -0.062, -0.059, ..., 0.064, 0.072, 0.149, 0.216, 0.610}
- Fidelity with respect to ideal state: 60.4%
- Check hypothesis: Can distribution of eigenvalues be explained by statistics and random matrix theory?

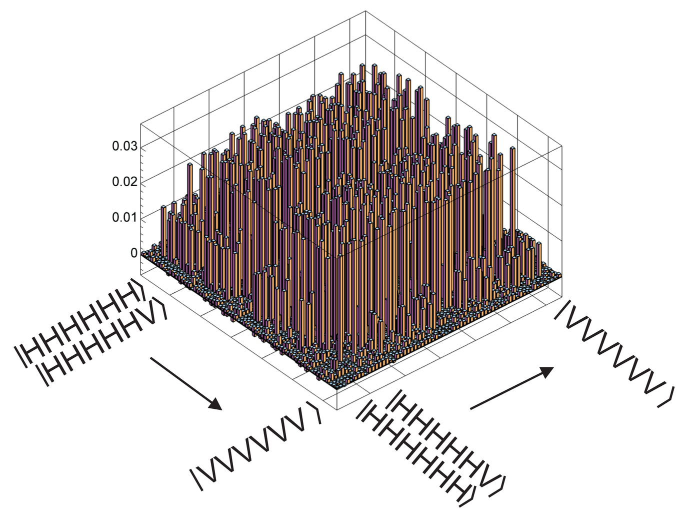
• Linearly reconstructed state

- Real part of density matrix



• Random Matrix Theory based Reconstruction

- Data of experimental six qubit Dicke state Real part of density matrix



Obtaining a physical state

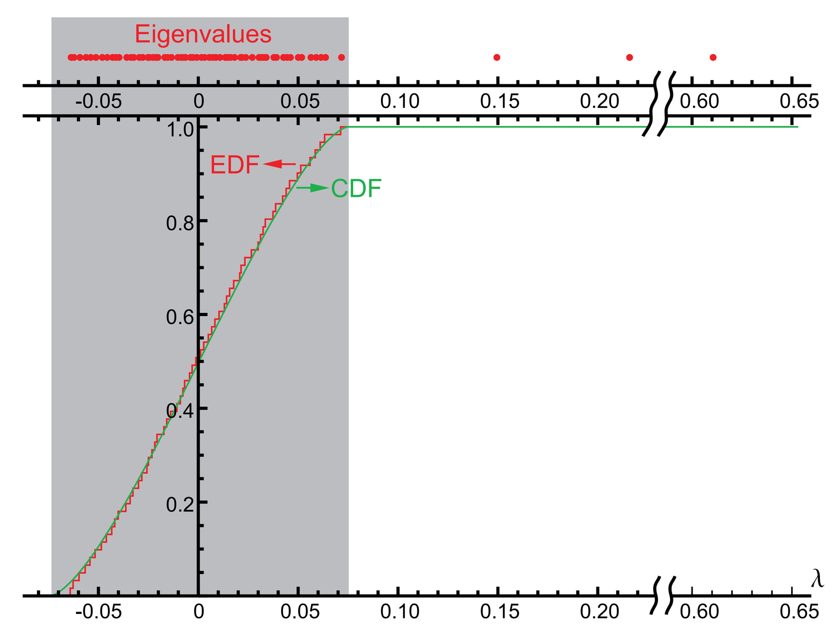
• Estimate the physical rank

- Can empirical distribution of eigenvalues be described by Wigner semicircle distribution?
- Hypotheses: *„smallest 2^n-r eigenvalues are noise and thus follow Wigner semicircle distribution"*
- Thus, no physical meaning attributed to low eigenvalues
- Determine p-values for quantifying validity of hypotheses
- use Anderson-Darling test
- For $r < 3$ eigenvalues outside of support of semicircle are found
- For $r > 3$ semicircle is centered around negative values
- Rejection of hypotheses for $r \neq 3$
- Detection of systematic deviations: state is not rank $r = 1$, but $r = 3$
- Obtain state by taking eigenstates of $r = 3$ largest eigenvalues and average rest of eigenvalues

- p-values for the first hypotheses

rank r	center c	radius R	P-value	P_{eff}-value
0	0.015625	0.076317	$9.44 \cdot 10^{-6}$	0
1	0.006187	0.075719	0.0089	0
2	0.002803	0.075115	0.3553	0
3	0.000399	0.074507	$1-8\cdot 10^{-7}$	$1-8\cdot 10^{-7}$
4	−0.000790	0.073894	0.9998	0
5	−0.001883	0.073275	0.9976	0

- empirical and cumulative distribution functions



References

[1] Z. Hradil, PRA 55, R1561 (1997); D. James, P. Kwiat, W. Munro, A. White, PRA 64, 052312 (2001)
[2] CS, LK, D. Richart, HW, T. Moroder, M. Kleinmann, O. Gühne, PRL 114, 080403 (2015)
[3] E. Wigner, Ann. Math. 62, 548 (1955); E. Wigner, Ann. Math. 67, 325 (1958)
[4] LK, CS, NK, JR, GT, HW, arXiv:1512.06866 (2015)
[5] M. Paris and J. Řeháček, Springer, *Quantum state estimation* (2004)
[6] CS, GT, A. Niggebaum, D. Gross, O. Gühne, HW, PRL 113, 040503 (2014)

Conclusions & Outlook

- Semicircle distribution describes spectrum of maximally mixed state in limit of many qubits
- Already applicable for only few qubits to describe noise spectrum in practical states
- Estimate necessary amount of data and probability for obtaining physical states
- Useful for detection of systematic deviations like misalignments or colored noise
- Applicable for state estimation by means of identifying deviations solely due to statistics